Choquet's Theorem for Constrained Sets of Quantum States

Seminari del Grup d'Informació Quàntica
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Abstract

Quantum technology is creating an increasing research activity in information theory, engineering, and physics. One example is the theory of quantum channels with input constraints. This line of research has seen new results in quantum communication theory and quantum dynamics, but its basic functional analysis is still unexplored. Here, we provide a version of Choquet's theorem regarding constrained sets of states. The result simplifies the definitions of several characteristics of quantum information theory.

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Information theory helps to optimize communication tasks:

Energy-Constraints

In order to quantify the capacity of a communication channel, one has to take into account a tradeoff between the energy expended and the communication achieved.

- T. M. Cover and J. A. Thomas, Elements of Information Theory, 1991
- A. S. Holevo, *On quantum communication channels with constrained inputs*, `arXiv:quant-ph/9705054` (1997)

Quantum States, Channels, Entropy

Let $\mathcal{H}$ be a separable Hilbert space and $\mathfrak{T}(\mathcal{H})$ the Banach space of trace-class operators.

States

In quantum information theory, a state is an element of the convex set of density operators

$$\mathfrak{S}(\mathcal{H}) = \{\rho \in \mathfrak{T}(\mathcal{H}) \mid \rho \succeq 0, \mathrm{Tr}(\rho) = 1\}.$$

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Channels

By channel $\Phi : A \to B$ we mean a linear, bounded, trace-preserving, completely positive map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$. Completely positive means that the map $\Phi \otimes \mathrm{Id}_n$ is positive for all $n = 1, 2, \ldots$.

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Entropy

The von Neumann entropy $H(\rho) \in [0, +\infty]$ of a state $\rho \in \mathfrak{S}(\mathcal{H})$ is the number $H(\rho) = -\operatorname{Tr}[\rho \log(\rho)]$.

The Classical Capacity of a Channel

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HSW Theorem (Holevo, Schumacher, Westmoreland)

Let $\dim(\mathcal{H}_A), \dim(\mathcal{H}_B) < \infty$. The classical capacity of Φ is the regularized Holevo information

$$C(\Phi) = \lim_{n\to\infty} \frac{1}{n}\chi(\Phi^{\otimes n}).$$

Here, the Holevo information of Φ is defined by

$$\chi(\Phi) = \sup_\pi \left\{ H[\Phi(\overline{\rho})] - \sum_i \pi_i H[\Phi(\rho_i)] \right\},$$

where $\overline{\rho} = \sum_i \pi_i \rho_i$ is the average state of the ensemble $\pi = \{\pi_i, \rho_i\}$.

The Classical Capacity of a Constrained Channel

The classical capacity $C(\Phi, F, E)$ of the constrained channel Φ is the asymptotically optimal rate at which classical information can be send through Φ if the input states ρ to the composite channel $\Phi^{\otimes n}$ satisfy the energy bound

$$\mathrm{Tr}(\rho F^{(n)}) \leq nE,$$

where $F^{(n)} = F \otimes I \otimes \cdots \otimes I + I \otimes F \otimes \cdots \otimes I + \ldots + I \otimes \cdots \otimes I \otimes F$, the operator F is discrete and positive (the energy observable), and $E > 0$.

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Theorem (Holevo '03, arXiv:quant-ph/0211170)

The classical capacity of the constrained channel Φ is

$$C(\Phi, F, E) = \lim_{n\to\infty} \frac{1}{n}\chi(\Phi^{\otimes n}, F^{(n)}, nE).$$

Here,

$$\chi(\Phi, F, E) = \sup_{\pi:\mathrm{Tr}\,\overline{\rho}F\leq E} \left\{ H[\Phi(\overline{\rho})] - \sum_i \pi_i H[\Phi(\rho_i)] \right\}.$$

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Gaussian encodings are optimal for certain constrained Gaussian channels describing the most common experimental realizations (Giovannetti et al. '14, arXiv:1312.2251)

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LSD Theorem (Lloyd, Shor, Devetak)

Let $\dim(\mathcal{H}_A), \dim(\mathcal{H}_B) < \infty$. The quantum capacity of Φ is the regularized coherent information

$$Q(\Phi) = \lim_{n \to \infty} \frac{1}{n} I_{\mathrm{c}}(\Phi^{\otimes n}).$$

Here, the coherent information of Φ is defined by

$$I_{\mathrm{c}}(\Phi) = \sup_{\rho \in \mathfrak{S}(\mathcal{H}_{\boldsymbol{A}})} \left[H(B)_\omega - H(RB)_\omega \right],$$

where $\omega_{RB} = \mathrm{Id}_R \otimes \Phi(\psi^\rho_{RA})$, and where $\psi^\rho_{RA} \in \mathfrak{S}(\mathcal{H}_{RA})$ is a purification of ρ with $\mathcal{H}_R \cong \mathcal{H}_A$.

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Theorem (Wilde, Qi '18, arXiv:1609.01997)

Let the energy observable F satisfy the Gibbs hypothesis $\mathrm{Tr}\, e^{-\theta F} < \infty$ for all $\theta > 0$, and let Φ have finite output entropy

$$\sup_{\rho \in \mathfrak{S}(\mathcal{H}_A) : \mathrm{Tr}\, \rho F \leq E} H[\Phi(\rho)] < \infty.$$

Then the quantum capacity of the constrained channel Φ is

$$Q(\Phi, F, E) = \lim_{n \to \infty} \frac{1}{n} I_{\mathrm{c}}(\Phi^{\otimes n}, F^{(n)}, nE).$$

Here, $I_{\mathrm{c}}(\Phi, F, E) = \sup_{\rho \in \mathfrak{S}(\mathcal{H}_A) : \mathrm{Tr}\, \rho F \leq E} \left[H(B)_\omega - H(RB)_\omega \right]$.

The ECD Norm

The diamond norm of a Hermitian-preserving linear map $\Theta : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ is defined as

$$\|\Theta\|_\diamond \doteq \sup_{\rho \in \mathfrak{S}(\mathcal{H}_{\boldsymbol{RA}})} \| \mathrm{Id}_R \otimes \Phi(\rho)\|_1$$

The minimum error probability of distinguishing channels Φ, Ψ is $\frac{1}{2}(1 - \|\Phi - \Psi\|_\diamond/2)$. The diamond norm generates a topology too strong for applications.

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Example (Winter '17, arXiv:1712.10267)

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ECD-Norm (Shirokov arXiv:1706.00361, Winter arXiv:1712.10267)

The energy-constrained diamond norm of $\Theta : A \to B$ is defined as

$$\|\Theta\|_{\diamond,E}^{F} \doteq \sup_{\rho \in \mathfrak{S}(\mathcal{H}_{\boldsymbol{RA}}) : \mathrm{Tr}\, \rho_{\boldsymbol{A}} F \leq E} \| \mathrm{Id}_R \otimes \Phi(\rho)\|_1 .$$

Continuity Bounds with Respect to the ECD Norm

A grounded Hamiltonian is a positive operator H on $\mathcal{H}$ such that

$$\inf\{\|H\varphi\| : \varphi \in \mathcal{D}(H), \|\varphi\| = 1\} = 0.$$

A channel $\Phi : A \to B$ is energy-limited with respect to grounded Hamiltonians H_A, H_B on $\mathcal{H}_A, \mathcal{H}_B$, if there are reals α, E_0 such that $\Phi^*(H_B) \preceq \alpha H_A + E_0$.

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Theorem (Winter '17, arXiv:1712.10267)

Let $\Phi, \Psi : A \to B$ be energy-limited channels with respect to H_A, H_B that satisfy the Gibbs hypothesis. If $\frac{1}{2}\|\Phi - \Psi\|_{\diamond,E}^{H_A} \leq \epsilon < 1$. Then

$$|C(\Phi, H_A, E) - C(\Psi, H_A, E)| \leq 56\delta H(\gamma_B(4\widetilde{E}/\delta)) + 6g(4\delta).$$

Here, $\delta = \sqrt{\epsilon}$, $\widetilde{E} = \alpha E + E_0$, $\gamma_B(\eta)$ is the Gibbs state with energy η, and $g(x) = (1+x)\log(1+x) - x\log(x)$.

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See the continuity bounds by Shirokov '17 (arXiv:1706.00361) and the follow-up paper on dynamical systems and capacities by Becker and Datta arXiv:1810.00863.

Generalized Compactness I

The success of energy constraints in quantum optics and mathematical physics drives our desire to know more about the functional analytic background.

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μ-Compactness

Let X be a closed, bounded subset of a separable Banach space and $M(X)$ the set of all Borel probability measures on X (weak topology); the barycenter of $\mu \in M(X)$ is

$$b(\mu) = \int_X x \, d\mu(x);$$

the set X is μ-compact if the pre-image of every compact subset of $\overline{\mathrm{conv}}(X)$ under $b : M(X) \to \overline{\mathrm{conv}}(X)$ is compact.

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Theorem 1 (Holevo, Shirokov '04, arXiv:quant-ph/0408176)

The set of density operators $\mathfrak{S}(\mathcal{H})$ is μ-compact.

Generalized Compactness II

Proposition 1 (Protasov and Shirokov '10, arXiv:1002.3610)

A closed subset of any μ-compact set is μ-compact.

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Theorem 2 (Protasov and Shirokov '10, arXiv:1002.3610)

Let C be a closed, bounded, μ-compact, convex set, which is a separable metric space. Then

$$C = \overline{\mathrm{conv}}(\mathrm{ext}\, C) \qquad \text{(Krein-Milman theorem)}$$

and

$$C = b\big(M(\,\overline{\mathrm{ext}\, C}\,)\big). \qquad \text{(Choquet theorem)}$$

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The principal topic of the paper by Protasov and Shirokov arXiv:1002.3610 is the generalization of the Vesterstrøm-O'Brien theory to μ-compact convex sets, and its application to quantum information theory.

Generalized Affine Constraints

Generalized Affine Maps

In the sequel, let V be a real vector space and $K \subseteq V$ a convex set. A generalized affine map on K is a map $f : K \to \mathbb{R} \cup \{+\infty\}$ that satisfies

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y). \qquad x, y \in K, \quad \lambda, \mu \geq 0, \quad \lambda + \mu = 1.$$

Let $\ell \in \mathbb{N}$, let f_k be a generalized affine map on K, and let $\alpha_k \in \mathbb{R}$ for all $k = 1, \ldots, \ell$. We define the sublevel set

$$K_\ell = \{x \in K : f_k(x) \leq \alpha_k \; \forall k = 1, \ldots, \ell\}.$$

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Expected Value Functional

Let H be an arbitrary positive operator on $\mathcal{H}$. Let $P_n = \int_0^n dE_H(\lambda)$ be the spectral projector of H corresponding to $[0, n]$, where E_H is the spectral measure of H. The expected value functional of H is the map defined by

$$\mathfrak{S}(\mathcal{H}) \to [0, +\infty], \qquad \rho \mapsto \operatorname{Tr} \rho H = \lim_{n\to\infty} \operatorname{Tr}(\rho H P_n).$$

Problem: Characterize Extreme Points

Consider several expected value functionals $f_1, f_2, \ldots, f_\ell : \mathfrak{S}(\mathcal{H}) \to [0, \infty]$.

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• The sublevel set $\mathfrak{S}(\mathcal{H})_\ell$ is closed, as the functionals $f_1, f_2, \ldots, f_\ell$ are lower semi-continuous. Proposition 1 and Theorem 1 and 2 show that any state $\rho \in \mathfrak{S}(\mathcal{H})_\ell$ can be represented as the barycenter $\rho = \int \sigma \mu(d\sigma)$ of some Borel probability measure μ supported by $\overline{\mathrm{ext}(\mathfrak{S}(\mathcal{H})_\ell)}$.

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• If the extreme points of $\mathfrak{S}(\mathcal{H})_\ell$ were a subset of the set of pure states $\mathrm{ext}\,\mathfrak{S}(\mathcal{H})$, then any state $\rho \in \mathfrak{S}(\mathcal{H})_\ell$ could be represented as the barycenter $\rho = \int \sigma \mu(d\sigma)$ of a Borel probability measure μ supported by the closed set of pure states

$$\mathrm{ext}(\mathfrak{S}(\mathcal{H})_\ell) = \mathfrak{S}(\mathcal{H})_\ell \cap \mathrm{ext}\,\mathfrak{S}(\mathcal{H}).$$

This would be a helpful simplification, as pure states are often easier to deal with.

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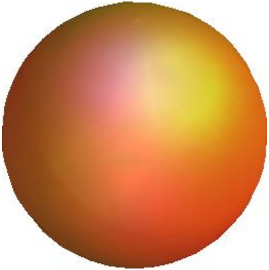
? Question

Under which conditions is every extreme point of $\mathfrak{S}(\mathcal{H})_\ell$ a pure state?

Affine Constraints on the Bloch Ball $\mathfrak{S}(\mathbb{C}^2)$



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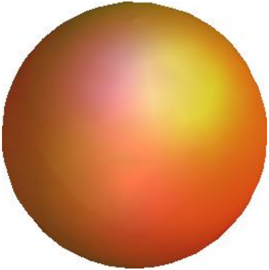

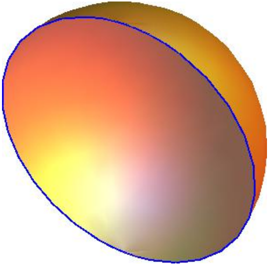

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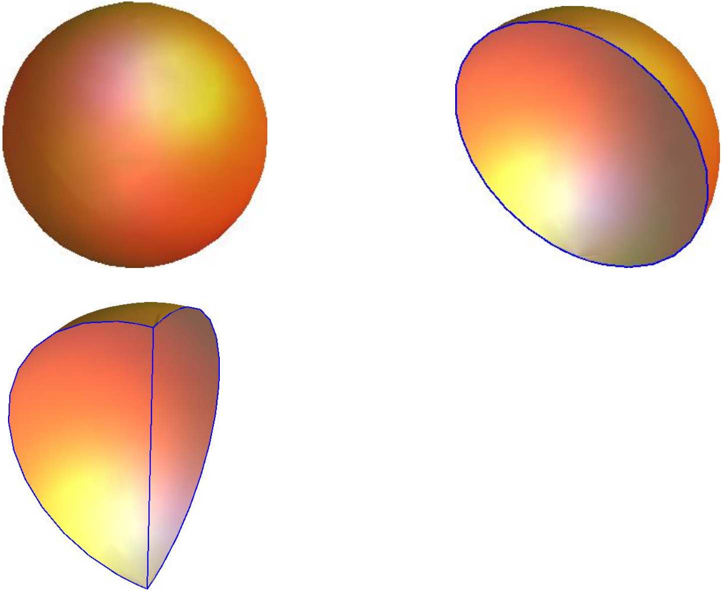

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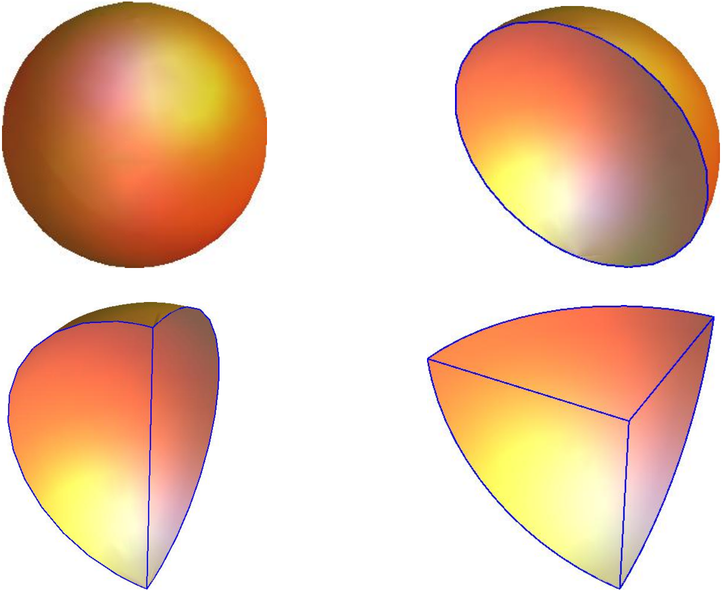

The Face Generated by a Point

Problem: Infinite-dimensional convex sets may have no "interior points" (A. Barvinok, *A Course in Convexity*, Providence, R.I: AMS, 2002). Not so for faces generated by points.

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Faces

A subset $E \subseteq K$ is an extreme subset of K if whenever $x = (1-\lambda)y + \lambda z$ lies in E for some $y, z \in K$ and $\lambda \in (0,1)$, then y and z are also in E. A face of K is a convex, extreme subset of K. The face of K generated by $x \in K$, denoted $F_K(x)$, is the intersection of all faces of K containing x.

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Theorem (W., Shirokov '20, arXiv:2003.14302)

Let V be a real vector space, $K \subseteq V$ a convex subset, and $x \in K$. Then

$$\operatorname{aff} F_K(x) = \big\{ y \in V \mid \exists \epsilon > 0 : x \pm \epsilon(y - x) \in F_K(x) \big\}.$$

In particular, x lies in the "interior" of $F_K(x)$.

The Face Generated by a Point

Problem: Infinite-dimensional convex sets may have no "interior points" (A. Barvinok, *A Course in Convexity*, Providence, R.I: AMS, 2002). Not so for faces generated by points.

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Proof. "$\supseteq$": Every open segment in $F_K(x)$ extends to a line in aff $F_K(x)$. Analyzing extreme subsets of $F_K(x)$, we prove "$\subseteq$" using the Kuratowski-Zorn lemma.

Corollaries to the Theorem

a) Let $x \in K$. Then $F_K(x) = \bigcup_{y,z \in K, x \in (y,z)} [y, z]$.

b) Let $E \subseteq K$ be a subset. The set E is an extreme subset of K if and only if the set E is a union of faces of K.

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Proposition 2 (W., Shirokov '20, arXiv:2003.14302)

Let $K, L \subseteq V$ be two convex sets and let $x \in K \cap L$. Then

- $F_{K \cap L}(x) = F_K(x) \cap F_L(x)$
- $\mathrm{ri}\left(F_K(x) \cap F_L(x)\right) = \mathrm{ri}\left(F_K(x)\right) \cap \mathrm{ri}\left(F_L(x)\right)$
- $\mathrm{aff}\left(F_K(x) \cap F_L(x)\right) = \mathrm{aff}\left(F_K(x)\right) \cap \mathrm{aff}\left(F_L(x)\right)$.

Example: Probability Distributions on $\mathbb{N}$

Let $\Delta_{\mathbb{N}} \doteq \left\{p : \mathbb{N} \to \mathbb{R} \mid p(n) \geq 0 \; \forall n \in \mathbb{N}, \text{ and } \sum_{n\in\mathbb{N}} p(n) = 1\right\}$ denote the σ-simplex of probability distributions on $\mathbb{N}$. Let $p \in \Delta_{\mathbb{N}}$ be a probability distribution with support I and let $\Delta_I \doteq \{q \in \Delta_{\mathbb{N}} \mid \mathrm{supp}(q) \subseteq I\}$.

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- Hadamard's trick: $p_H(n) = p(n)/(\sqrt{r_n} + \sqrt{r_{n+1}}) = \sqrt{r_n} - \sqrt{r_{n+1}}$, where $r_n = \sum_{m\geq n} p(m)$ for all $n \in \mathbb{N}$, is a probability distribution with support I and

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- The set $\Delta_{\mathbb{N}}$ has a continuous chain of faces: Let $p_s(n) = \zeta(s)^{-1} \cdot n^{-s}$, where $\zeta(s) = \sum_{n\in\mathbb{N}} n^{-s}$ is the Euler-Riemann zeta function, $s > 1$. Then p_s is a probability distribution with support $\mathbb{N}$ and $F_{\Delta_{\mathbb{N}}}(p_s) \subseteq F_{\Delta_{\mathbb{N}}}(p_t) \iff s \geq t$.

Face Dimensions of Sublevel Sets

Using Proposition 2, we show that by replacing the convex set K with its sublevel set K_1, the dimension of the face generated by a point may decrease by at most one.

Theorem

Let x be a point in K_1. If the face $F_{K_1}(x)$ of K_1 generated by x has dimension $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, then the face $F_K(x)$ of K generated by x has dimension m or $m + 1$.

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Iterating the theorem, we can exploit gaps in the face dimensions of K.

Corollary

Let K have no face with dimension $1, \ldots, \ell$. Then every extreme point of the sublevel set K_ℓ is an extreme point of K.

Theorems for two Expected Value Constraints I

The face dimensions of the set $\mathfrak{S}(\mathcal{H})$ of density operators is a number from the list $0, 3, 8, \ldots, n^2 - 1, \ldots, \infty$.

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Krein-Milman and Choquet Theorem

Assume the sublevel set $\mathfrak{S}(\mathcal{H})_2$ is nonempty. Then the set of extreme points $\mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)$ is nonempty and closed and is equal to the set of pure states in $\mathfrak{S}(\mathcal{H})_2$.

- The set $\mathfrak{S}(\mathcal{H})_2$ is the closure of the convex hull of $\mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)$.

- Any state $\rho \in \mathfrak{S}(\mathcal{H})_2$ is the barycenter $\rho = \int \sigma\mu(d\sigma)$ of some Borel probability measure μ supported by $\mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)$.

Theorems for two Expected Value Constraints II

Theorem 3 (Maximizing Convex Functions)

- Let $f : \mathfrak{S}(\mathcal{H})_2 \to [-\infty, \infty]$ be a convex function on the sublevel set $\mathfrak{S}(\mathcal{H})_2$. If f is either lower semicontinuous or upper semicontinuous and upper bounded, then

$$\sup\{f(\rho) : \rho \in \mathfrak{S}(\mathcal{H})_2\} = \sup\{f(\rho) : \rho \in \mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)\}. \tag{1}$$

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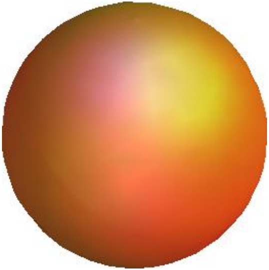
• If f is upper semicontinuous, and if one of the positive operators defining the two expected value functionals has discrete spectrum of finite multiplicity, then $\mathfrak{S}(\mathcal{H})_2$ is compact and the supremum on the right-hand side of Equation (1) is attained at a pure state in $\mathfrak{S}(\mathcal{H})_2$.

Affine Constraints on the Bloch Ball $\mathfrak{S}(\mathbb{C}^2)$



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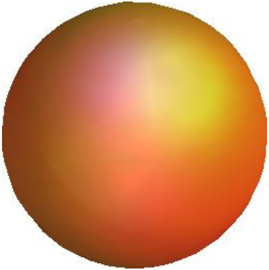

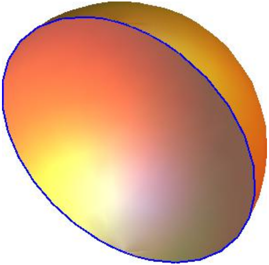

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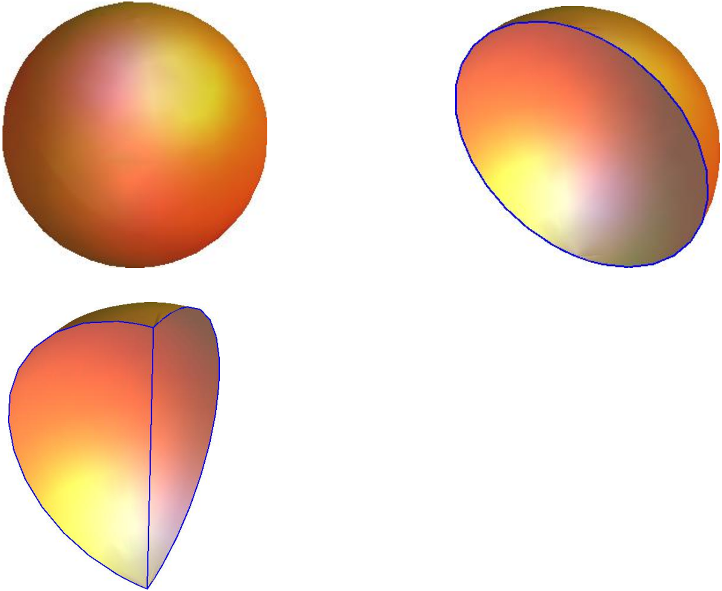

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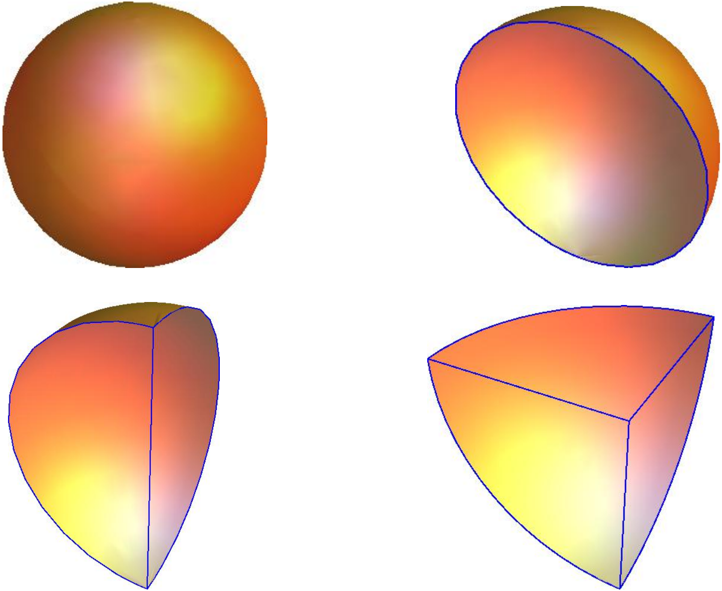

Applications: Minimal Output Entropy I

Minimal Output Entropy

The minimal output entropy of a channel $\Phi : A \to B$ is defined as

$$H_{\min}(\Phi) = \inf_{\rho \in \mathfrak{S}(\mathcal{H}_A)} H(\Phi(\rho)) = \inf_{\varphi \in \mathcal{H}_{A,1}} H(\Phi(|\varphi\rangle\langle\varphi|)),$$

where $\mathcal{H}_{A,1}$ is the unit sphere in $\mathcal{H}_A$.

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The additivity of the minimal output entropy, which means that

$$H_{\min}(\Phi \otimes \Psi) = H_{\min}(\Phi) + H_{\min}(\Psi)$$

for all channels Φ and Ψ, was disproved by Hasting ('08, arXiv:0809.3972). This is important, because the additivity of $H_{\min}$ is equivalent to the additivity of the Holevo information (Shor '03, arXiv:quant-ph/0305035). The additivity of the Holevo information would imply that the classical capacity is $C(\Phi) = \chi(\Phi)$, without the regularization $C(\Phi) = \lim_{n\to\infty} \frac{1}{n}\chi(\Phi^{\otimes n})$.

Applications: Minimal Output Entropy II

Constrained Minimal Output Entropy

In studies of an infinite-dimensional channels $\Phi : A \to B$, it is reasonable to consider the constrained minimal output entropy

$$H_{\min}(\Phi, G, E) = \inf_{\rho \in \mathfrak{S}(\mathcal{H}_A) : \operatorname{Tr} \rho G \leq E} H(\Phi(\rho)), \tag{2}$$

where G is a positive operator, the energy observable.

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• In finite dimensions, the infimum in (2) can be taken only over pure states satisfying the condition $\operatorname{Tr} \rho G \leq E$ (Memarzadeh, Mancini '16, arXiv:1605.04525). The above Theorem 3 proves the same assertion for an arbitrary infinite-dimensional channel Φ and for any energy observable G.

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• $H_{\min}(\hat{\Phi}, G, E) = H_{\min}(\Phi, G, E)$ holds for every complementary channel $\hat{\Phi}$, as $H[\hat{\Phi}(\rho)] = H[\Phi(\rho)]$ holds for all pure states (Section 8.3.4 in A. S. Holevo, *Quantum Systems, Channels, Information*, Berlin: De Gruyter, 2019).

Applications: Operator E-Norms I

The KSW-Theorem (Kretschmann, Schlingemann, Werner '07, arXiv:0710.2495) shows that the Stinespring representation is continuous.

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Stinespring Theorem

Given a completely positive linear map $\Phi : A \to B$, there exists a Hilbert space $\mathcal{H}_R$ and an operator $V_\Phi : \mathcal{H}_A \to \mathcal{H}_{RB}$ such that $\Phi(\rho) = \mathrm{Tr}_R\, V_\Phi \rho V_\phi^*$ for all $\rho \in \mathfrak{T}(\mathcal{H}_A)$.

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Let G be a grounded Hamiltonian with dense domain. Given $E > 0$, the operator E-norm of $A \in \mathfrak{B}(\mathcal{H}_A, \mathcal{H}_{RB})$ is

$$\|A\|_E^G \doteq \sup_{\rho \in \mathfrak{S}(\mathcal{H}_A) : \operatorname{Tr} \rho G \leq E} \sqrt{\operatorname{Tr} A \rho A^*}.$$

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Corollary

Theorem 3 shows that the operator E-norm is a constrained version of the operator norm, that is to say, $\|A\|_E^G = \sup_{\varphi \in \mathcal{H}_{\boldsymbol{A},\mathbf{1}}, \langle \varphi | G | \varphi \rangle \leq E} \|A\varphi\|$.

Applications: Operator E-Norms II

Theorem (Shirokov '18, arXiv:1806.05668)

For any completely positive linear maps Φ and Ψ from $\mathfrak{T}(\mathcal{H}_A)$ to $\mathfrak{T}(\mathcal{H}_B)$, the following inequalities hold.

$$\frac{\|\Phi - \Psi\|_{\diamond,E}^{G}}{\sqrt{\|\Phi\|_{\diamond,E}^{G}} + \sqrt{\|\Psi\|_{\diamond,E}^{G}}} \leq \inf_{V_\Phi, V_\Psi} \|V_\Phi - V_\Psi\|_E^G \leq \sqrt{\|\Phi - \Psi\|_{\diamond,E}^{G}},$$

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The theorem improves the original KSW-theorem, which uses the unconstrained diamond norm on the space of Hermitian-preserving linear maps and the unconstrained operator norm on the set of Stinespring operators.

Thank you for your attention!

These slides were created with LaTeX (beamer class and bclogo-package). The graphics were drawn with `Wolfram Mathematica`.