Choquet's Theorem for Constrained Sets of Quantum States

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Abstract

Experimental quantum information processing presents challenging problems to the theory, because of input constraints to quantum channels and because the convex set of quantum states is not compact. Here, we explain a version of Choquet's theorem regarding constrained sets of states. This analytic tool considerably simplifies the definitions of several characteristics of quantum information theory employed in the context of the additivity of channel capacities and in the continuity of Stinespring dilations.

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Quantum States, Channels, Entropy

Let $\mathcal{H}$ be a separable Hilbert space and $\mathfrak{T}(\mathcal{H})$ the Banach space of trace-class operators.

States

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$$\mathfrak{S}(\mathcal{H}) = \{\rho \in \mathfrak{T}(\mathcal{H}) \mid \rho \succeq 0, \mathrm{Tr}(\rho) = 1\}.$$

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By channel we mean a linear, bounded, trace-preserving, completely positive map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$. Completely positive means that the map $\Phi \otimes \mathrm{Id}_n$ is positive for all $n = 1, 2, \dots$.

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Entropy

The von Neumann entropy $H(\rho) \in [0, +\infty]$ of a state $\rho \in \mathfrak{S}(\mathcal{H})$ is the number $H(\rho) = -\mathrm{Tr}[\rho \log(\rho)]$.

The Classical Capacity of a Channel

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HSW Theorem (Holevo, Schumacher, Westmoreland)

Let $\dim(\mathcal{H}) < \infty$. The classical capacity of Φ is the regularized Holevo information

$$C(\Phi) = \lim_{n\to\infty} \frac{1}{n}\chi(\Phi^{\otimes n}).$$

Here, the Holevo information of Φ is defined by

$$\chi(\Phi) = \sup_{\pi} \left\{ H[\Phi(\overline{\rho})] - \sum_i \pi_i H[\Phi(\rho_i)] \right\},$$

where $\overline{\rho} = \sum_i \pi_i \rho_i$ is the average state of the ensemble $\pi = \{\pi_i, \rho_i\}$.

The Classical Capacity of a Constrained Channel

Let $F = \sum_j f_j |e_j\rangle\langle e_j|$ be an energy observable, where $\{|e_j\rangle\}$ is an ONB in $\mathcal{H}$, $\{f_j\}$ a sequence of nonnegative reals, and let $E > 0$. The classical capacity $C(\Phi, F, E)$ of the constrained channel Φ is the asymptotically optimal rate at which classical information can be send through Φ if the input states ρ to the composite channel $\Phi^{\otimes n}$ satisfy the energy bound $\mathrm{Tr}(\rho F^{(n)}) \leq nE$, where $F^{(n)} = F \otimes I \otimes \cdots \otimes I + \ldots + I \otimes \cdots \otimes I \otimes F$.

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Theorem (Holevo '03, arXiv:quant-ph/0211170)

The classical capacity of the constrained channel Φ is

$$C(\Phi, F, E) = \lim_{n\to\infty} \frac{1}{n} \chi(\Phi^{\otimes n}, F^{(n)}, nE).$$

Here,

$$\chi(\Phi, F, E) = \sup_{\pi : \mathrm{Tr}\, \overline{\rho} F \leq E} \left\{ H[\Phi(\overline{\rho})] - \sum_i \pi_i H[\Phi(\rho_i)] \right\}.$$

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• Gaussian encodings provide the optimal rates for certain constrained Gaussian channels which describe the most common experimental realizations of quantum communication (Giovannetti, Holevo, García-Patrón '14, arXiv:1312.2251).

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LSD Theorem (Lloyd, Shor, Devetak)

Let $\dim(\mathcal{H}) < \infty$. The quantum capacity of Φ is the regularized coherent information

$$Q(\Phi) = \lim_{n\to\infty} \frac{1}{n} I_{\mathrm{c}}(\Phi^{\otimes n}).$$

Here, the coherent information of Φ is defined by

$$I_{\mathrm{c}}(\Phi) = \sup_{\rho\in\mathfrak{S}(\mathcal{H}_{\boldsymbol{A}})} \left[H(B)_\omega - H(RB)_\omega\right],$$

where $\omega_{RB} = \mathrm{Id}_R \otimes \Phi(\psi^\rho_{RA})$, and where $\psi^\rho_{RA} \in \mathfrak{S}(\mathcal{H}_{RA})$ is a purification of ρ with $\mathcal{H}_R \cong \mathcal{H}_A$.

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Theorem (Wilde, Qi '18, arXiv:1609.01997)

Let the energy observable F satisfy $\mathrm{Tr}\, e^{-\theta F} < \infty$ for all $\theta > 0$ and let Φ have finite output entropy

$$\sup_{\rho \in \mathfrak{S}(\mathcal{H}_A) : \mathrm{Tr}\, \rho F \leq E} H[\Phi(\rho)] < \infty.$$

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$$Q(\Phi, F, E) = \lim_{n \to \infty} \frac{1}{n} I_{\mathrm{c}}(\Phi^{\otimes n}, F^{(n)}, nE).$$

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$$Q(\Phi, F, E) = \lim_{n \to \infty} \frac{1}{n} I_c(\Phi^{\otimes n}, F^{(n)}, nE).$$

Here, $I_c(\Phi, F, E) = \sup_{\rho \in \mathfrak{S}(\mathcal{H}_A) : \operatorname{Tr} \rho F \leq E} \left[H(B)_\omega - H(RB)_\omega \right]$.

- The capacities $C(\Phi, F, E)$ and $Q(\Phi, F, E)$ are continuous in Φ with respect to the energy-constrained diamond norms $\|\Theta\|_\diamond^E = \sup_{\sigma \in \mathfrak{S}(\mathcal{H}_{RA}) : \operatorname{Tr} \rho F \leq E} \| \mathrm{Id}_R \otimes \Theta(\rho) \|_1$, called ECD-norms (Shirokov '17 arXiv:1706.00361, Winter '17 arXiv:1712.10267).

Generalized Compactness I

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μ-Compactness

Let X be a closed, bounded subset of a separable Banach space and $M(X)$ the set of all Borel probability measures on X (weak topology); the barycenter of $\mu \in M(X)$ is

$$b(\mu) = \int_X x \, d\mu(x);$$

the set X is μ-compact if the pre-image of every compact subset of $\overline{\mathrm{conv}}(X)$ under $b : M(X) \to \overline{\mathrm{conv}}(X)$ is compact.

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Theorem 1 (Holevo, Shirokov '04, arXiv:quant-ph/0408176)

The set of density operators $\mathfrak{S}(\mathcal{H})$ is μ-compact.

Generalized Compactness II

Proposition 1 (Protasov, Shirokov '10, arXiv:1002.3610)

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Let C be a closed, bounded, μ-compact, convex set, which is a separable metric space. Then

$$C = \overline{\mathrm{conv}}(\mathrm{ext}\, C) \qquad \text{(Krein-Milman theorem)}$$

and

$$C = b\big(M(\overline{\mathrm{ext}\, C})\big). \qquad \text{(Choquet theorem)}$$

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- The principal topic of the paper arXiv:1002.3610 is the generalization of the Vesterstrøm-O'Brien theory to μ-compact convex sets, and its application to quantum information theory.

Generalized Affine Constraints

Generalized Affine Maps

In the sequel, let V be a real vector space and $K \subseteq V$ a convex set. A generalized affine map on K is a map $f : K \to \mathbb{R} \cup \{+\infty\}$ that satisfies

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y). \qquad x, y \in K, \quad \lambda, \mu \geq 0, \quad \lambda + \mu = 1.$$

Let $\ell \in \mathbb{N}$, let f_k be a generalized affine map on K, and let $\alpha_k \in \mathbb{R}$ for all $k = 1, \ldots, \ell$. We define the sublevel set

$$K_\ell = \{x \in K : f_k(x) \leq \alpha_k \; \forall k = 1, \ldots, \ell\}.$$

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Expected Value Functional

Let H be an arbitrary positive operator on $\mathcal{H}$. Let $P_n = \int_0^n dE_H(\lambda)$ be the spectral projector of H corresponding to $[0, n]$, where E_H is the spectral measure of H. The expected value functional of H is the map defined by

$$\mathfrak{S}(\mathcal{H}) \to [0, +\infty], \qquad \rho \mapsto \operatorname{Tr} \rho H = \lim_{n \to \infty} \operatorname{Tr}(\rho H P_n).$$

Problem: Characterize Extreme Points

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• The sublevel set $\mathfrak{S}(\mathcal{H})_\ell$ is closed, as the functionals $f_1, f_2, \ldots, f_\ell$ are lower semi-continuous. Proposition 1 and Theorem 2 above show that any state $\rho \in \mathfrak{S}(\mathcal{H})_\ell$ can be represented as the barycenter $\rho = \int \sigma \mu(d\sigma)$ of some Borel probability measure μ supported by $\overline{\mathrm{ext}(\mathfrak{S}(\mathcal{H})_\ell)}$.

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• If the extreme points of $\mathfrak{S}(\mathcal{H})_\ell$ were a subset of the set of pure states $\mathrm{ext}\,\mathfrak{S}(\mathcal{H})$, then any state $\rho \in \mathfrak{S}(\mathcal{H})_\ell$ could be represented as the barycenter $\rho = \int \sigma \mu(d\sigma)$ of a Borel probability measure μ supported by the closed set of pure states

$$\mathrm{ext}(\mathfrak{S}(\mathcal{H})_\ell) = \mathfrak{S}(\mathcal{H})_\ell \cap \mathrm{ext}\,\mathfrak{S}(\mathcal{H}).$$

This would be a helpful simplification, as pure states are often easier to deal with.

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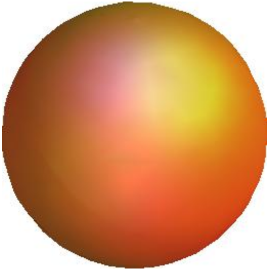
? Question

Under which conditions is every extreme point of $\mathfrak{S}(\mathcal{H})_\ell$ a pure state?

Affine Constraints on the Bloch Ball $\mathfrak{S}(\mathbb{C}^2)$



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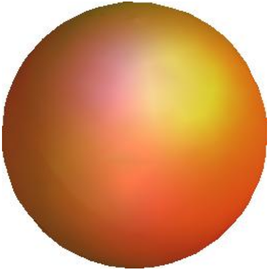

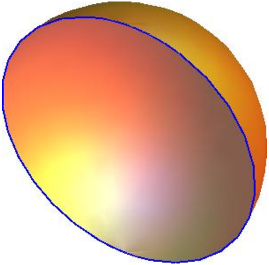

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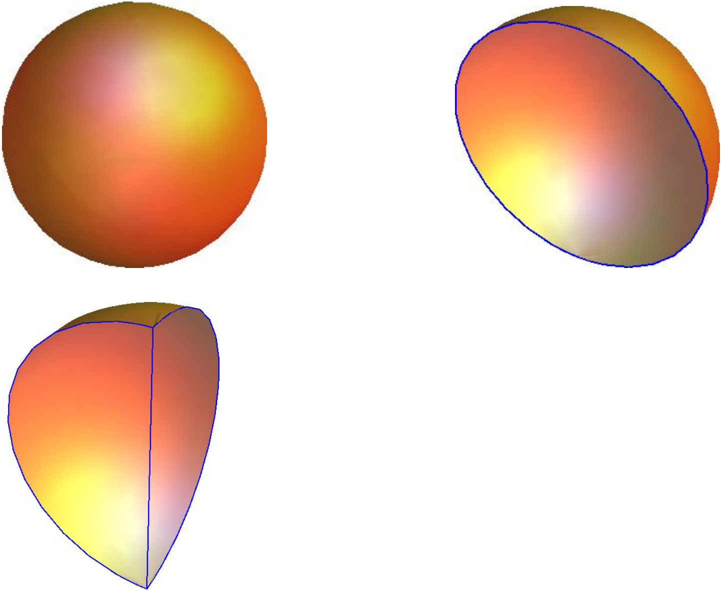

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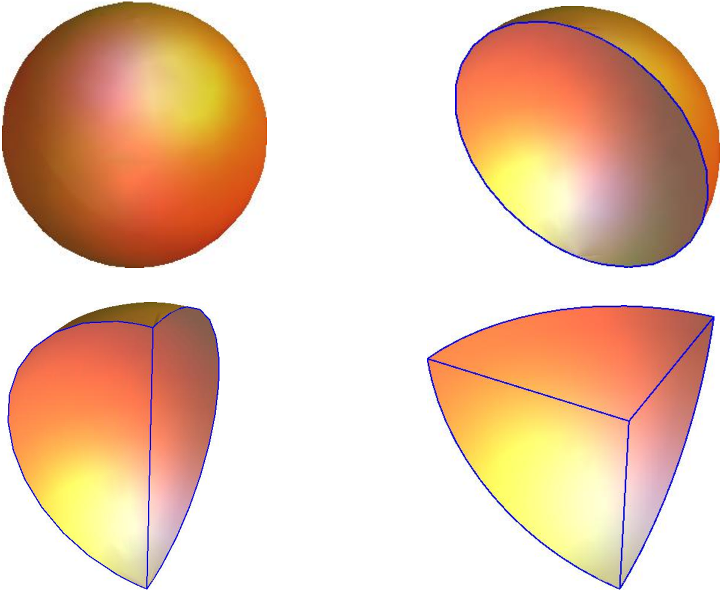

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Faces

A subset $E \subseteq K$ is an extreme subset of K if whenever $x = (1 - \lambda)y + \lambda z$ lies in E for some $y, z \in K$ and $\lambda \in (0, 1)$, then y and z are also in E. A face of K is a convex, extreme subset of K. The face of K generated by $x \in K$, denoted $F_K(x)$, is the intersection of all faces of K containing x.

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Theorem 3 (W., Shirokov '20, arXiv:2003.14302)

Let V be a real vector space, $K \subseteq V$ a convex subset, and $x \in K$. Then

$$\operatorname{aff} F_K(x) = \big\{ y \in V \mid \exists \epsilon > 0 : x \pm \epsilon(y - x) \in F_K(x) \big\}.$$

In particular, x lies in the "interior" of $F_K(x)$.

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Proof. "$\supseteq$" is easy to see. "$\subseteq$" follows from the Kuratowski-Zorn lemma and from analyzing extreme subsets of $F_K(x)$.

Corollaries to Theorem 3

a) Let $x \in K$. Then $F_K(x) = \bigcup_{y,z \in K, x \in (y,z)} [y, z]$.

b) Let $E \subseteq K$ be a subset. The set E is an extreme subset of K if and only if the set E is a union of faces of K.

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Proposition 2

Let $K, L \subseteq V$ be two convex sets and let $x \in K \cap L$. Then

- $F_{K \cap L}(x) = F_K(x) \cap F_L(x)$
- $\operatorname{ri}\left(F_K(x) \cap F_L(x)\right) = \operatorname{ri}\left(F_K(x)\right) \cap \operatorname{ri}\left(F_L(x)\right)$
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Example: Probability Distributions on $\mathbb{N}$

Let $p \in \Delta_{\mathbb{N}} \doteq \left\{p : \mathbb{N} \to \mathbb{R} \mid p(n) \geq 0 \; \forall n \in \mathbb{N}, \text{ and } \sum_{n \in \mathbb{N}} p(n) = 1\right\}$,
let $I = \operatorname{supp}(p)$ be the support of p, and let $\Delta_I \doteq \{q \in \Delta_{\mathbb{N}} \mid \operatorname{supp}(q) \subseteq I\}$.

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- Let $I \subseteq \mathbb{N}$ be infinite. Then $\mathrm{conv}\left(\{\delta_n \mid n \in I\}\right) \subset F_{\Delta_{\mathbb{N}}}(p) \subset \Delta_I$ are three different faces of $\Delta_{\mathbb{N}}$.

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- Corollary a) shows $F_{\Delta_{\mathbb{N}}}(p) = \{q \in \Delta_I \mid \sup_{n \in I} q(n)/p(n) < \infty\}$.

- If I is finite, then $F_{\Delta_{\mathbb{N}}}(p) = \operatorname{conv}(\{\delta_n \mid n \in I\}) = \Delta_I$.

- Let $I \subseteq \mathbb{N}$ be infinite. Then $\operatorname{conv}(\{\delta_n \mid n \in I\}) \subset F_{\Delta_{\mathbb{N}}}(p) \subset \Delta_I$ are three different faces of $\Delta_{\mathbb{N}}$.

- Hadamard's trick: Let $p_H(n) = p(n)/(\sqrt{r_n} + \sqrt{r_{n+1}}) = \sqrt{r_n} - \sqrt{r_{n+1}}$, where $r_n = \sum_{m \geq n} p(m)$ for all $n \in \mathbb{N}$. Then p_H is a probability distribution with support I and $F_{\Delta_{\mathbb{N}}}(p) \subset F_{\Delta_{\mathbb{N}}}(p_H) \subset F_{\Delta_{\mathbb{N}}}((p_H)_H) \subset \cdots \subset \Delta_I$.

Example: Probability Distributions on $\mathbb{N}$

Let $p \in \Delta_{\mathbb{N}} \doteq \left\{p : \mathbb{N} \to \mathbb{R} \mid p(n) \geq 0 \ \forall n \in \mathbb{N}, \text{ and } \sum_{n\in\mathbb{N}} p(n) = 1\right\}$,
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- The set $\Delta_{\mathbb{N}}$ has a continuous chain of faces: Let $p_s(n) = \zeta(s)^{-1} \cdot n^{-s}$, where $\zeta(s) = \sum_{n\in\mathbb{N}} n^{-s}$ is the Euler-Riemann zeta function, $s > 1$. Then p_s is a probability distribution with support $\mathbb{N}$ and $F_{\Delta_{\mathbb{N}}}(p_s) \subseteq F_{\Delta_{\mathbb{N}}}(p_t) \iff s \geq t$.

Face Dimensions of Sublevel Sets

Using Proposition 2, we show that by replacing the convex set K with its sublevel set K_1, the dimension of the face generated by a point may decrease by at most one.

Theorem 4

Let x be a point in K_1. If the face $F_{K_1}(x)$ of K_1 generated by x has dimension $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, then the face $F_K(x)$ of K generated by x has dimension m or $m + 1$.

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Iterating Theorem 4, we can exploit gaps in the face dimensions of K.

Corollary 1

Let K have no face with dimension $1, \ldots, \ell$. Then every extreme point of the sublevel set K_ℓ is an extreme point of K.

Theorems for two Expected Value Constraints I

The face dimensions of the set $\mathfrak{S}(\mathcal{H})$ of density operators is a number from the list $0, 3, 8, \ldots, n^2 - 1, \ldots, \infty$.

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Krein-Milman and Choquet Theorem

Assume the sublevel set $\mathfrak{S}(\mathcal{H})_2$ is nonempty. Then the set of extreme points $\mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)$ is nonempty and closed and is equal to the set of pure states in $\mathfrak{S}(\mathcal{H})_2$.

- The set $\mathfrak{S}(\mathcal{H})_2$ is the closure of the convex hull of $\mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)$.
- Any state $\rho \in \mathfrak{S}(\mathcal{H})_2$ is the barycenter $\rho = \int \sigma\mu(d\sigma)$ of some Borel probability measure μ supported by $\mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)$.

Theorems for two Expected Value Constraints II

Theorem 6 (Maximizing Convex Functions)

- Let $f : \mathfrak{S}(\mathcal{H})_2 \to [-\infty, \infty]$ be a convex function on the sublevel set $\mathfrak{S}(\mathcal{H})_2$. If f is either lower semicontinuous or upper semicontinuous and upper bounded, then

$$\sup\{f(\rho) : \rho \in \mathfrak{S}(\mathcal{H})_2\} = \sup\{f(\rho) : \rho \in \mathrm{ext}(\mathfrak{S}(\mathcal{H})_2)\}. \tag{1}$$

Theorems for two Expected Value Constraints II

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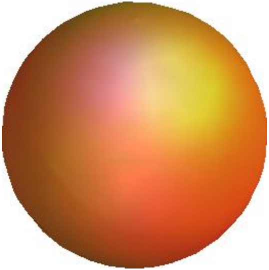
• If f is upper semicontinuous, and if one of the positive operators defining the two expected value functionals has discrete spectrum of finite multiplicity, then $\mathfrak{S}(\mathcal{H})_2$ is compact and the supremum on the right-hand side of Equation (1) is attained at a pure state in $\mathfrak{S}(\mathcal{H})_2$.

Affine Constraints on the Bloch Ball $\mathfrak{S}(\mathbb{C}^2)$



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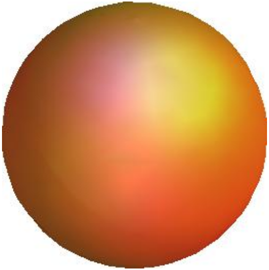

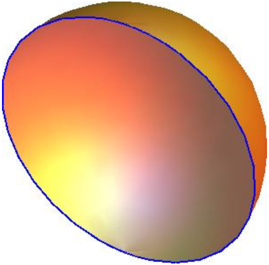

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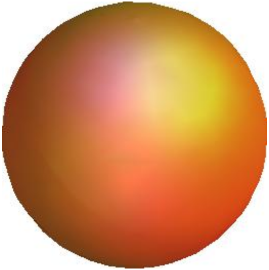

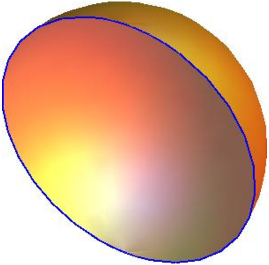

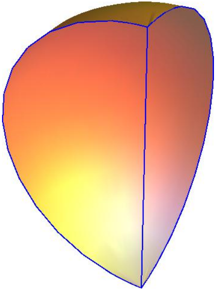

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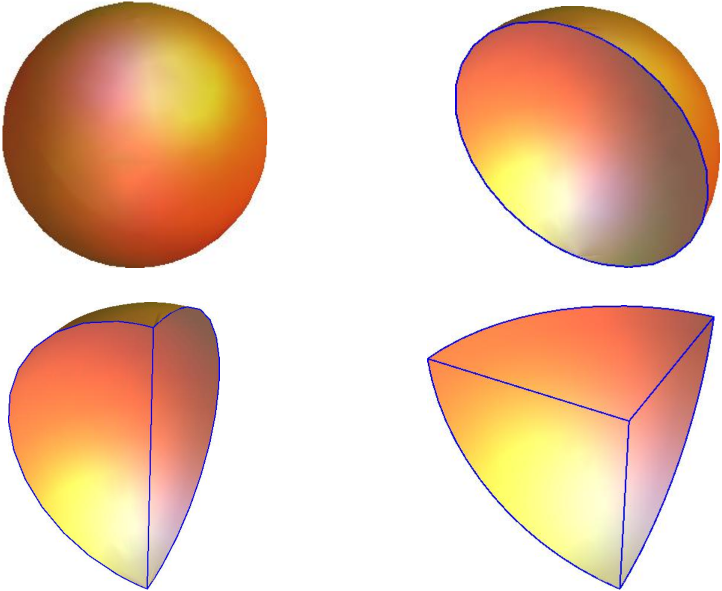

Applications: Minimal Output Entropy I

Minimal Output Entropy

The minimal output entropy of a channel $\Phi : A \to B$ is defined as

$$H_{\min}(\Phi) = \inf_{\rho \in \mathfrak{S}(\mathcal{H}_A)} H(\Phi(\rho)) = \inf_{\varphi \in \mathcal{H}_{A,1}} H(\Phi(|\varphi\rangle\langle\varphi|)),$$

where $\mathcal{H}_{A,1}$ is the unit sphere in $\mathcal{H}_A$.

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where $\mathcal{H}_{A,1}$ is the unit sphere in $\mathcal{H}_A$.

The additivity of the minimal output entropy, which means that

$$H_{\min}(\Phi \otimes \Psi) = H_{\min}(\Phi) + H_{\min}(\Psi)$$

for all channels Φ and Ψ, was disproved by Hasting ('08, arXiv:0809.3972). This is important, because the additivity of $H_{\min}$ is equivalent to the additivity of the Holevo information (Shor '03, arXiv:quant-ph/0305035). The additivity of the Holevo information would imply that the classical capacity is $C(\Phi) = \chi(\Phi)$, without the regularization $C(\Phi) = \lim_{n\to\infty} \frac{1}{n}\chi(\Phi^{\otimes n})$.

Applications: Minimal Output Entropy II

Constrained Minimal Output Entropy

In studies of an infinite-dimensional channels $\Phi : A \to B$, it is reasonable to consider the constrained minimal output entropy

$$H_{\min}(\Phi, G, E) = \inf_{\rho \in \mathfrak{S}(\mathcal{H}_A) : \operatorname{Tr} \rho G \leq E} H(\Phi(\rho)), \tag{2}$$

where G is a positive operator, the energy observable.

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• In finite dimensions, the infimum in (2) can be taken only over pure states satisfying the condition $\operatorname{Tr} \rho G \leq E$ (Memarzadeh, Mancini '16, arXiv:1605.04525). The above Theorem 6 proves the same assertion for an arbitrary infinite-dimensional channel Φ and for any energy observable G.

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• $H_{\min}(\hat{\Phi}, G, E) = H_{\min}(\Phi, G, E)$ holds for every complementary channel $\hat{\Phi}$, as $H[\hat{\Phi}(\rho)] = H[\Phi(\rho)]$ holds for all pure states (Section 8.3.4 in A. S. Holevo, *Quantum Systems, Channels, Information*, Berlin: De Gruyter, 2019).

Applications: Operator E-Norms I

The operator E-norms appear in the generalization (Shirokov '18, arXiv:1806.05668) of the KSW-Theorem (Kretschmann, Schlingemann, Werner '07, arXiv:0710.2495), which shows that the Stinespring representation of completely positive linear maps is continuous.

Stinespring Theorem

Given a completely positive linear map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$, there exists a Hilbert space $\mathcal{H}_R$ and an operator $V_\Phi : \mathcal{H}_A \to \mathcal{H}_{RB}$ such that $\Phi(\rho) = \mathrm{Tr}_R\, V_\Phi \rho V_\phi^*$ for all $\rho \in \mathfrak{T}(\mathcal{H}_A)$.

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• Let G be a positive semidefinite operator on $\mathcal{H}_A$ with dense domain $\mathcal{D}(G)$ such that $\inf\{\|G\varphi\| : \varphi \in \mathcal{D}(G), \|\varphi\| = 1\} = 0$. Given $E > 0$, the operator E-norm of $A \in \mathfrak{B}(\mathcal{H}_A, \mathcal{H}_{RB})$ is

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- Theorem 6 shows that $\|A\|_E^G = \sup_{\varphi \in \mathcal{H}_{A,1}, \langle \varphi | G | \varphi \rangle \leq E} \|A\varphi\|$, a constrained version of the operator norm.

Applications: Operator E-Norms II

• The energy-constrained cb-norm of a Hermitian-preserving linear map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ is defined as

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For any completely positive linear maps Φ and Ψ from $\mathfrak{T}(\mathcal{H}_A)$ to $\mathfrak{T}(\mathcal{H}_B)$, the following inequalities hold.

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where the infimum is over all common Stinespring representations.

• The theorem improves the original KSW-theorem, which uses the unconstrained cb-norm on the space of Hermitian-preserving linear maps and the unconstrained operator norm on the set of Stinespring operators.

Thank you for your attention!

These slides were created with LaTeX (beamer class and bclogo-package). The graphics were drawn with `Wolfram Mathematica`.