Invariants for the Ideal Boundary of a Tree

Invarianten für den Idealen Rand eines Baumes

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Vorwort

Diese Arbeit war ursprünglich geplant als eine Erweiterung um ein Kapitel zur Masters Arbeit „Dynamics on Graphs“ [Wei04], vorgelegt im April 2004 an der School of Mathematics, University of Bristol, Großbritannien. Die Gesamtarbeit sollte als Diplomarbeit eingereicht werden am Mathematischen Institut der Friedrich-Alexander Universität Erlangen-Nürnberg.

Während der Entwicklung stellte sich heraus, daß viele Begriffe vereinfacht oder zusammengefaßt werden konnten. So hat die Arbeit nun aus ästhetischen Gründen einen eigenen unabhängigen Aufbau, der ausgelegt ist, um den geometrischen Zugang transparenter werden zu lassen. Die Sprache Englisch wurde beibehalten um dem Forschungsaspekt zu dienen, unter dem auch die vorangegangene Masters Arbeit stand. Diese war innerhalb des internationalen Forschungsnetzwerkes „Mathematical Aspects of Quantum Chaos“ entstanden, das einen bedeutenden Knoten in Bristol hat.

Ganz herzlich möchte ich mich bei Professor Andreas Knauf bedanken für die sehr gute Betreuung. Bei ihm fand ich jederzeit ein offenes Ohr für meine Anliegen und Probleme und konnte seine konstante Unterstützung und Motivation wahrnehmen.

Trotz der subtropischen Wetterverhältnisse kurz vor der Abgabe fanden sich Agata, Christoph und Doris bereit, die Arbeit korrekturzulesen. Dafür möchte ich ihnen mein herzlichstes Dankeschön sagen.

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Chapter 1

Introduction

Methods from the analysis of geodesic flows on Riemannian manifolds of negative curvature are applied to the study of dynamics on a finite connected graph. In particular, it turns out that there is the notion of *cross ratio* and *geodesic delay* for a tree. The deeper reason for this lies in the fact that all (locally finite, connected) graphs are non-positively curved spaces in the sense of Alexandrov (see [BBI01], Ch. 4.2).

1.1 Riemannian Manifolds and Graphs

The geodesic flow $\phi_t : \mathrm{T}^1M \to \mathrm{T}^1M$ $(t \in \mathbb{R})$ on a compact connected n-dimensional Riemannian manifold (M, g) of strictly negative sectional curvature is ergodic[1]. This fact has first been proved by D. Anosov in 1967. (See [Bal95] for a more recent proof by M. Brin.) The much simpler case of *constant* negative sectional curvature was treated by E. Hopf in [Hop39].

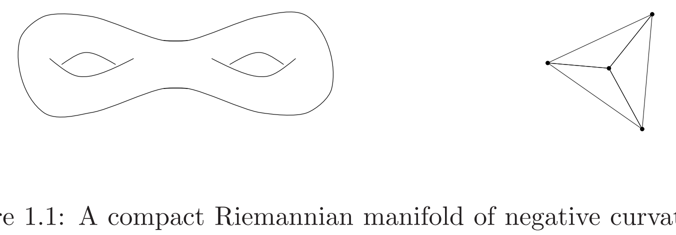


Figure 1.1: A compact Riemannian manifold of negative curvature (left) — a finite connected graph (right).

The dynamics on a finite connected graph that is described by transitions from a vertex to some adjacent vertex along the time $\mathbb{Z}$ is ergodic[1] if (and only if) the graph is not a circuit where backtracking is allowed only in one direction. Such a dynamics is mixing[1], if (and only if) there are two closed orbits that have coprime lengths. As an easy example, the tetrahedron depicted in Figure 1.1 has a mixing dynamics. See [Wei04] for a proof of these statements.

1.2 The Ideal Boundary

Following U.Hamenstädt [Ham99], the *space of geodesics* $\mathcal{G}\widetilde{M}$ on the universal cover $(\widetilde{M}, \widetilde{g})$ of a (compact, connected) Riemannian manifold (M, g) of dimension n with strictly negative sectional curvature is defined as the set of unit-speed flow lines modulo parameterization:

$$\mathcal{G}\widetilde{M} \quad := \quad \left\{ c : \mathbb{R} \mapsto \mathrm{T}^1\widetilde{M} \quad : \quad \begin{array}{c} c \text{ is a solution of the} \\ \text{geodesic equations} \end{array} \right\} / \, \mathbb{R}. \tag{1.1}$$

Using unique existence of the geodesic c with $c(0) \in \mathrm{T}^1\widetilde{M}$, this space can naturally be identified with the quotient of the unit-tangent bundle $\mathrm{T}^1\widetilde{M}$ under the action of the geodesic flow $\Phi_t : \mathrm{T}^1\widetilde{M} \to \mathrm{T}^1\widetilde{M}$ $(t \in \mathbb{R})$

$$\mathcal{G}\widetilde{M} \quad = \quad \mathrm{T}^1\widetilde{M} / \, \mathbb{R}. \tag{1.2}$$

See e.g. [KS02] for a presentation of the concept of the geodesic flow.

The *ideal boundary* of a Riemannian manifold is defined intrinsically as the set of equivalence classes of unit-speed ray asymptotics, see e.g. [Bal95]. As the universal cover $\widetilde{M}$ is diffeomorphic to the open ball $\widetilde{D} := \{x \in \mathbb{R}^n : |x| < 1\}$, the ideal boundary of the Riemannian manifold $(\widetilde{M}, \widetilde{g})$ can be identified with the sphere $S^{n-1} = \partial\widetilde{D}$. The space of geodesics has a natural identification with pairs of distinct boundary points

$$\mathcal{G}\widetilde{M} \quad = \quad \big(\partial\widetilde{M} \times \partial\widetilde{M}\big) - \Delta, \tag{1.3}$$

$\Delta := \big\{(\gamma, \gamma) : \gamma \in \partial\widetilde{M}\big\}$ being the diagonal.

[1] A measure preserving transformation $T : X \to X$ on a probability space $(X, \mathcal{B}, m)$ is called *ergodic*, if $T^{-1}(B) = B$ for $B \in \mathcal{B}$ implies $m(B) \in \{0, 1\}$. It is called *mixing*, if for all $A, B \in \mathcal{B}$ holds $\lim_{n\to\infty} m\big(T^{-n}(A) \cap B\big) = m(A)m(B)$. Analogue definitions are set for flows.

As an example we consider the Poincaré disk $\widetilde{D} := \{z \in \mathbb{C} : |z| < 1\}$ with hyperbolic metric as a Riemannian manifold of constant negative curvature -1 and $\mathbb{R}$-dimension $n = 2$ (see e.g. [BV86]). The ideal boundary can be identified with the topological boundary $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ of $\widetilde{D}$. Elements of the geodesic space $\mathcal{G}\widetilde{D}$ are oriented circular arcs perpendicular to the boundary, including oriented straight lines through the origin $0 \in \mathbb{C}$. The correspondence of Eqn. (1.3) in this context is given by assigning to each orbit of the geodesic flow the ordered pair $(c^-, c^+) \in (S^1 \times S^1) - \Delta$ with $c^\pm := \lim_{t \to \pm\infty} c(t)$ for the geodesic $c : \mathbb{R} \to \widetilde{D}$.

Observe that in Riemannian geometry there are local objects like Christoffel symbols and a curvature tensor. With $(\widetilde{M}, \widetilde{g})$ as above, global statements like equations (1.2) and (1.3) are proved *analytically.* A tree however consists of isolated vertices, so that tools from infinitesimal calculus are not applicable. To escape from this dilemma, we use methods from *synthetic* geometry: a space of curves is defined axiomatically on some space of points and makes a metric space out of this point set (when there is a curve linking each two points). Many more concepts like angles and curvature are defined for such spaces. The recent progress on this field of mathematics is described, e.g. in [BBI01].

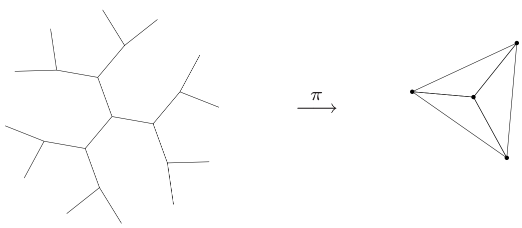


Figure 1.2: Projection from a tree to a covered graph.

The dynamics on a finite connected graph can be investigated using methods from synthetic geometry, too. The curves linking vertices are paths. The dynamics is lifted from the graph to a tree, since there is a sensible class of paths for geometric considerations. These paths are called *geodesics* and they are characterized by the fact that they have no backtracking, indeed they are shortest paths. A method for constructing a covering tree $\mathcal{T}$ for a graph A was presented in the Master's thesis [Wei04]. See also J.-P. Serre [Ser80] and

H. Bass [Bas93] for original literature. The tree $\mathcal{T}$ is wrapped over the graph A by the projection $\pi : \mathcal{T} \to A$; confer Figure 1.2.

An abstract and more general description of this situation is the following: we consider a tree $\mathcal{T}$ with vertex set $\mathrm{V}(\mathcal{T})$ and a group of isometries acting upon; and we consider the space $\mathcal{G}$ of bi-infinite geodesics $g : \mathbb{Z} \to \mathcal{T}$ of $\mathcal{T}$ with the shift operator L acting by a left-shift on each geodesic. Note, if $\mathcal{G}$ is non-empty then $\mathcal{T}$ is infinite because geodesics are injective paths.

The *ideal boundary* $\mathcal{T}(\infty)$ of $\mathcal{T}$ is defined as the set of equivalence classes of asymptotic (right-) rays. More precisely, the equivalence is a condition on infinite intersection. The *velocity space*

$$\mathcal{V}_{\mathcal{T}} \quad := \quad \big(\mathcal{T}(\infty) \times \mathcal{T}(\infty)\big) - \Delta, \tag{1.4}$$

$\Delta = \big\{(\gamma, \gamma) : \gamma \in \mathcal{T}(\infty)\big\}$ denoting the diagonal, is by Theorem 1 in the correspondence

$$\mathcal{V}_{\mathcal{T}} \quad = \quad \mathcal{G}/\mathbb{Z}. \tag{1.5}$$

On the other hand, for each vertex $z \in \mathrm{V}(\mathcal{T})$ a so-called unit tangent space $\mathrm{T}^1_z\mathcal{T} := \big\{l \in \mathcal{V}_{\mathcal{T}} : z \in l\big\}$ is defined. This set consists of all "directions" that a geodesic through z can assume. The so-called unit tangent bundle is defined as

$$\mathrm{T}^1\mathcal{T} := \Big\{ \quad (z, l) \quad : \quad l \in \mathrm{T}^1_z\mathcal{T}, \quad z \in \mathrm{V}(\mathcal{T}) \quad \Big\}.$$

By Theorem 9 the velocity space can also be identified with a quotient of the unit tangent bundle:

$$\mathcal{V}_{\mathcal{T}} \quad = \quad \mathrm{T}^1\mathcal{T}/\mathbb{Z}. \tag{1.6}$$

1.3 Patterson-Sullivan Construction

The *hyperbolic space* $\mathbb{H}^n$ is diffeomorphic to the open ball $\widetilde{D} := \{x \in \mathbb{R}^n : |x| < 1\}$. The ideal boundary of $\mathbb{H}^n$, while intrinsically defined, is naturally identified with the sphere $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. A geometrical construction that involves *conformal densities*[2], on the ideal boundary can be used to write invariant measures for the geodesic flow on the unit tangent bundle $\mathrm{T}^1\mathbb{H}^n$, confer [Sul79].

[2] These densities are called α-dimensional densities in [BM96]

M. Burger and S. Mozes [BM96] extended this construction to very general spaces, that include also the case of a tree. They showed in particular that a measure that is constructed in this way on the universal cover of a finite connected graph A corresponds to a so-called Markov measure for a symbolic dynamics defined on the edges of A.

It was shown in the Master's thesis [Wei04], that in case of ergodicity, these measures resemble the Parry measure if the covering tree for the graph A is minimal ([Wei04], Prop. 5.25). The Parry measure has an interpretation as the asymptotic distribution of periodic orbits; in the case that the dynamics is mixing, it is the unique measure of maximal entropy [KH89].

1.4 Symplectic Structure of a Tree

The central aim of this diploma thesis is to transcribe concepts and methods from geodesic flows of Riemannian manifolds to trees. We consult the reference U. Hamenstädt [Ham99] again.

1.1 Definition. A *generalized cross ratio* is a Hölder continuous positive function Cr on the space of quadruples of pairwise distinct points in $\partial\widetilde{M}$ with the following properties:

1. Cr is invariant under the action of the fundamental group $\pi_1(M)$ on $\left(\partial\widetilde{M}\right)^4$;
2. $Cr(\xi,\xi',\eta,\eta') = Cr(\xi',\xi,\eta,\eta')^{-1}$;
3. $Cr(\xi,\xi',\eta,\eta') = Cr(\eta,\eta',\xi,\xi')$;
4. $Cr(\xi,\xi',\eta,\eta')Cr(\xi',\xi'',\eta,\eta') = Cr(\xi,\xi'',\eta,\eta')$;
5. $Cr(\xi,\xi',\eta,\eta')Cr(\xi',\eta,\xi,\eta')Cr(\eta,\xi,\xi'\eta') = 1$.

We will find in Section 10.2 a function $[\cdot] : \mathcal{T}_{\mathrm{Q}}^{\infty} \to \mathbb{Z}$ called (oriented) *concordance* on the space $\mathcal{T}_{\mathrm{Q}}^{\infty}$ of quadrilaterals of boundary points such that $e^{[\cdot]}$ behaves algebraically like a cross ratio.

Further, Hamenstädt states in [Ham99] that there is a special and distinguished cross ratio, on mutually distinct quadruples of elements of $\partial\widetilde{M}$. It is called the *cross ratio of the metric* $\widetilde{g}$ or more precisely the *cross ratio of the* *length cocycle of the metric* $\widetilde{g}$. She continues to say "this cocycle can be viewed as an integrated version of the symplectic form $\mathrm{d}\omega$ on $\mathcal{G}\widetilde{M}$".

Whether the above cross ratio on $\mathcal{T}_{\mathrm{Q}}^{\infty}$ is analogous to this special cocycle remains an open question for future research.

Outside of this context but not less surprisingly, we show in Section 10.3, that the concordance $[\cdot]$ behaves on the unit tangent bundle $\mathrm{T}^1\mathcal{T}$ as a symmetric bilinear form on the tangent bundle $\mathrm{T}\widetilde{M}$ does in the restriction to the unit tangent bundle $\mathrm{T}^1\widetilde{M}$.

1.5 Overview of this Thesis

Chapter 2 explains the graph model which is taken from J.P. Serre [Ser80] and which is probably more general than necessary, because we are only working with trees here. This model formed the basis of the Master's thesis [Wei04] and has been absorbed from there.

Chapter 3 introduces *lines* as the most basic geometric concept (besides a vertex). We use the same way of writing lines as it is done in [FTN91]. In contrast to that reference, a unifying way of writing geodesic segments, geodesic right-rays, geodesic left-rays and bi-infinite with the same syntax is formally developed. On the way through the chapter, the ideal boundary is defined and the identification of $\mathcal{V}_{\mathcal{T}} = \mathcal{G}/\mathbb{Z}$ is proved. Whereas these chapters mainly review known results, my own work starts in **Chapter 4**.

There *Triangles* are defined; and — probably more important — a bifurcation as the unique vertex in the intersection of the three sides of a triangle is identified. There are a few very simple rules proved for handling bifurcations.

Chapter 5 is about *quadrilaterals*. This is the climax of geometrical examinations. (Have a look at the sizes of the following two geometrical chapters and compare to this one.) The two important invariant quantities *Klein type* and *inner diameter* of a quadrilateral are discussed exhaustively. The *concordance* $[\cdot]$ is defined as a function from the space of quadrilaterals to $\mathbb{Z}$ and many relations are proved for this function.

Chapter 6 is a short trip to *pentagons* only to prove one more equation for the concordance (which is a function on quadrilaterals).

Chapter 7 introduces the geometrical concept of a *horocycle* centered at some boundary point. Horocycles allow to compare the distances of two vertices to boundary points (which are infinitely far away!).

Chapter 8 examines the *geodesic space* $\mathcal{G}$ of a tree in greater detail. The identification of $\mathcal{G} = \mathcal{V}_{\mathcal{T}} \times \mathbb{Z}$ is endowed with a particular parameterization κ_x that is fixed by choosing a base point x in the tree. *Unstable-* and *stable manifolds* are defined and operations on these manifolds are expressed in coordinates. The *unit tangent bundle* $\mathrm{T}^1\mathcal{T}$ is defined and the identification $\mathcal{V}_{\mathcal{T}} = \mathrm{T}^1\mathcal{T}/\mathbb{Z}$ is proved.

Chapter 9 We investigate a vector space of invariant functions on $\mathcal{T}_{\mathrm{Q}}^{\infty}$ available in every tree — among them are *concordance* and *delay* — and we check their symmetry behavior under permutations of variables. There is a function that comprises the full information about Klein type and inner diameter. Thus we can prove that there is only one independent invariant function on $\mathcal{T}_{\mathrm{Q}}^{\infty}$ for the case of a regular tree and the full automorphism group. In the last section, the delay is shown to have a geometrical interpretation as a *geodesic delay*.

Chapter 10 starts with a first glance at the problem of finding invariant functions for specific pairs $(\mathcal{T}, H)$ of a tree $\mathcal{T}$ and a group $H < \mathrm{Is}(\mathcal{T})$. More important than this topic is to show how the results of this thesis can be consulted to approach some of the questions of Section 1.4. This will be done in the last two sections.

Chapter 2

Trees

2.1 Graphs and their Morphisms

A *graph* Γ in the sense of Serre [Ser80] consists of a set $X = \mathrm{V}(\Gamma)$, a set $Y = \mathrm{E}(\Gamma)$ and the *graph maps*

$$\begin{array}{rclcrcl} Y & \longrightarrow & X \times X, & & e & \longmapsto & (\mathrm{o}(e), \mathrm{t}(e)) \\ Y & \longrightarrow & Y, & & e & \longmapsto & \overline{e}, \end{array}$$

which satisfy for all $e \in Y$

$$\overline{\overline{e}} = e, \quad \overline{e} \neq e \quad \text{and} \quad \mathrm{o}(e) = \mathrm{t}(\overline{e}). \tag{2.1}$$

We will often write Γ instead of $\mathrm{V}(\Gamma)$ when no confusion is possible. The maps o and t are called *point maps*. An element $x \in X$ is called a *vertex* of the graph Γ, an element $e \in Y$ is called an *(oriented) edge* and $\overline{e}$ is called the *inverse edge to* e. The map $^-$ is an involution on the set of edges Y, so that its orbits provide

Figure 2.1: The borders of an edge e (left). The edge $\overline{e}$ (right) points into the opposite direction and has the same borders.

a partition of the edge set Y into subsets $\{e, \overline{e}\}$ each of which has two edges.

Such a set including an edge together with its inverse is called a *geometric edge*. The vertex $\mathrm{o}(e) = \mathrm{t}(\bar{e})$ is called the *origin* of e and the vertex $\mathrm{t}(e) = \mathrm{o}(\bar{e})$ is called the *terminus* of e. These two vertices are called the *borders* of e. Two vertices are called *adjacent*, if they are the borders of some edge e. Figure 2.1 displays the borders of an edge e and of the inverse edge $\overline{e}$.

We form for every vertex $x \in \mathrm{V}\Gamma$ the set $\mathrm{St}^{\Gamma}(x) = \{e \in \mathrm{E}\Gamma : \mathrm{o}(e) = x\}$, the *star* at x and write simply $\mathrm{St}(x)$. If $\mathrm{St}(x)$ is finite, then its cardinality is called the *degree* of x, in short $\deg(x)$. Otherwise we set $\deg(x) = \infty$. If $\deg(x)$ is finite for all vertices, Γ is called *locally finite*. We are only interested in locally finite graphs. A graph is called *finite*, when it has a finite number of edges and vertices. If $\deg(x) = k$ for all vertices $x \in \mathrm{V}\Gamma$ then Γ is called *regular* or more specifically k*-regular*.

There are maps, called *graph morphisms* between graphs that preserve the graph structure. Given two graphs Γ_1 and Γ_2, a function

$$F : \begin{cases} \mathrm{V}(\Gamma_1) & \longrightarrow & \mathrm{V}(\Gamma_2) \\ \mathrm{E}(\Gamma_1) & \longrightarrow & \mathrm{E}(\Gamma_2) \end{cases}$$

is a morphism, if it obeys for all $e \in \mathrm{E}\Gamma_1$ the rules

$$\begin{aligned} F\big(\mathrm{o}(e)\big) &= \mathrm{o}\big(F(e)\big) \\ F\big(\overline{e}\big) &= \overline{F(e)}. \end{aligned} \tag{2.2}$$

The map F will be denoted simply by $F : \Gamma_1 \to \Gamma_2$.

A morphism F is called injective, surjective or bijective, if the restrictions of F to the set of vertices $\mathrm{V}(\Gamma_1)$ and to the set of edges $\mathrm{E}(\Gamma_1)$ have these properties. If $\Gamma_2 = \Gamma_1$ then the morphism F is called an *endomorphism*. A bijective morphism is called an *isomorphism*, a bijective endomorphism is called an *automorphism* and we define $\mathrm{Aut}(\Gamma)$ as the set of all automorphisms of Γ. Note that the $\mathrm{Aut}(\Gamma)$ is a group under composition because a composition of morphisms is a morphism and the inverse map to an isomorphism is an isomorphism, see [Wei04].

A morphism α from a graph Γ_1 to a graph Γ_2 is *locally injective*, *locally surjective* respectively *locally bijective* if the restriction

$$\begin{array}{cccc} \alpha_x : & \mathrm{St}^{\Gamma_1}(x) & \longrightarrow & \mathrm{St}^{\Gamma_2}(\alpha(x)) \\ & \bigcap & & \bigcap \\ & \mathrm{E}\Gamma_1 & & \mathrm{E}\Gamma_2 \end{array}$$

of α to the *local map* $\alpha_x = \alpha|_{\mathrm{St}(x)}$ is injective, surjective respectively bijective for each $x \in X$.

2.2 Diagrams

Graphs are represented pictorially in accordance with the following convention: a point marked on the diagram corresponds to a vertex of the graph, a line joining two marked points corresponds to a geometric edge.

We draw an arrow instead of a line in diagrams of graphs, if we want to refer to an edge rather than to a geometrical edge with some label. Sometimes, when drawing large graphs, we do not draw vertices (i.e. we remove them after drawing the lines).

2.1 Example (Graphs and diagrams)**.**

- A graph with one vertex x and two edges e,$\overline{e}$ is represented by each of the diagrams in Figure 2.2.

Figure 2.2: Diagrams with one point and one line

- A graph with two distinct vertices x, y and two edges $e, \overline{e}$ that have borders x, y is represented by each of the diagrams in Figure 2.3. The first

Figure 2.3: Diagrams with two points joined by a line

diagram does not specify if x or y is the origin of e. The second diagram does.

2.3 Paths, Circuits and Trees

An *orientation* of a graph Γ is a subset $Y^+ \subset \mathrm{E}(\Gamma)$ such that $\mathrm{E}(\Gamma) = Y^+ \sqcup \overline{Y^+}$. The edges of Y^+ are called *positive* edges. A graph with an orientation is called an *oriented graph*. Note that an oriented graph is fully specified by a set of vertices X, the orientation Y^+ and information about origin and terminus for the edges of Y^+, see [Wei04]. The edges in $\overline{Y^+}$ are in one-to-one correspondence to the ones of Y^+ and their origin and terminus vertices are given by equation (2.1). In diagrams of oriented graphs, always the positive edge e of each geometrical edge is drawn as an arrow from the origin $\mathrm{o}(e)$ to the terminus $\mathrm{t}(e)$.

The oriented graph path_n has vertices $\{0, 1, \dots, n\}$ $(n \geq 1)$. The orientation is given by the n edges $[i, i+1]$, $0 \leq i < n$, where $\mathrm{o}([i, i+1]) = i$ and $\mathrm{t}([i, i+1]) = i + 1$ (see Figure 2.4).

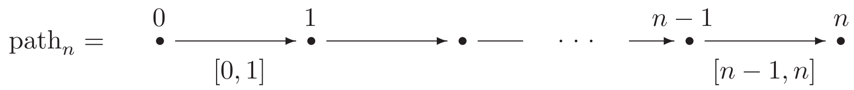


Figure 2.4: The oriented graph path_n

A morphism p from path_n to a graph Γ is called a *path*, the natural number n is called the *length* of p and will be denoted as $\mathrm{len}(p)$. The *edge sequence* of a path p of length n is given by

$$p([0, 1]), \dots, p([n-1, n]),$$

These edges satisfy $\mathrm{t}(p[i, i+1]) = \mathrm{o}(p[i+1, 1+2])$ according to the morphism rules (2.2).

Conversely, if a sequence $(e_1, \dots, e_n)$ of $n \geq 1$ edges in a graph Γ is given such that $\mathrm{t}(e_i) = \mathrm{o}(e_{i+1})$ holds for all $1 \leq i < n$, then a morphism $p : \mathrm{path}_n \to \Gamma$ is defined through $F([i, i+1]) := e_{i+1}$ on the positive edges of path_n. Equation (2.2) extends p to the vertices of path_n and to the remaining edges of p in a consistent way. Thus we can define a path p through the edge sequence

$$p = (e_1, \dots, e_n). \tag{2.3}$$

For a path p of length n the sequence of vertices

$$p(0), \ldots, p(n)$$

is called the *vertex sequence* of p. Two consecutive vertices of the vertex sequence of a path are adjacent. For completeness one should mention also paths of length zero. The graph path_0 consists of a single vertex and no edges. A path of length zero is represented by a vertex in a graph and has an empty edge sequence.

A path p of length $n \geq 0$ is called a path *from* $p(0)$ *to* $p(n)$, the path p *joins* $p(0)$ with $p(n)$. A graph Γ is called *connected*, if each two vertices $x, y \in \mathrm{V}(\Gamma)$ are joined by a path. On the vertex set of each connected graph Γ, a metric can be defined. The distance between two vertices $x, y \in \mathrm{V}(\Gamma)$ is given by the length of the shortest path joining x with y:

$$\mathrm{d}(x, y) := \min \Big\{ \quad \mathrm{len}(p) \quad : \quad p \text{ joins } x \text{ with } y \quad \Big\}. \tag{2.4}$$

The function $\mathrm{d} : \mathrm{V}(\Gamma) \times \mathrm{V}(\Gamma) \to \mathbb{N}_0$ defines a metric on $\mathrm{V}(\Gamma)$, cp. [Wei04].

2.2 Definition (Isometries)**.** An *isometry* from a graph Γ to a graph Γ' is a morphism $h : \Gamma \to \Gamma'$ such that d is invariant under h, that is $\mathrm{d}\big(h(x), h(y)\big) = \mathrm{d}(x, y)$ for all vertices $x, y \in \mathrm{V}(\Gamma)$. By $\mathrm{Is}\,(\Gamma)$ we denote the set of isometries $h : \Gamma \to \Gamma$. Not that every automorphism $h \in \mathrm{Aut}\,(\Gamma)$ is an isometry of Γ.

2.3 Lemma. *Isometries* $h : \Gamma \to \Gamma'$ *are injective on the vertex set* $\mathrm{V}(\Gamma)$.

Proof. This follows from the fact that d is a metric. If $h(x) = h(y)$ for two vertices x, y, then $\mathrm{d}(x, y) = \mathrm{d}\big(h(x), h(y)\big) = 0$ and therefore $x = y$. □

Infinite paths will be used, too. There is the one-sided version path_∞, which has vertices $\{0, 1, 2, \ldots\}$. The orientation is given by the edges $[i, i+1]$, $i \in \mathbb{N}_0$, where $\mathrm{o}([i, i+1]) = i$ and $\mathrm{t}([i, i+1]) = i + 1$ (compare Figure 2.5). A morphism p from path_∞ to a graph Γ is called a *ray* or more specifically a *right ray*. The graph $\mathrm{path}_{-\infty}$ has vertices $\{\ldots, -2, -1, 0\}$. The orientation is given by the edges $[i, i+1]$, $i \in -\mathbb{N}$, where $\mathrm{o}([i, i+1]) = i$ and $\mathrm{t}([i, i+1]) = i + 1$ (compare Figure 2.6). A morphism p from $\mathrm{path}_{-\infty}$ to a graph Γ is called a *ray* or more specifically a *left ray*. There is the two-sided version $\mathcal{T}_2$, with vertices $\mathbb{Z}$. The orientation is given by the edges $[i, i+1]$, $i \in \mathbb{Z}$, where $\mathrm{o}([i, i+1]) = i$

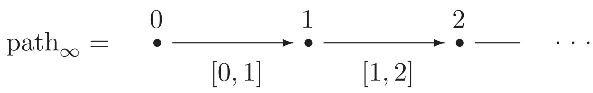


Figure 2.5: The oriented graph path_∞

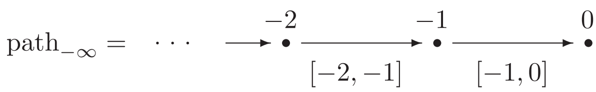


Figure 2.6: The oriented graph $\text{path}_{-\infty}$

and $\text{t}([i, i+1]) = i + 1$ (confer Figure 2.7). A morphism p from $\mathcal{T}_2$ to a graph Γ is called a *bi-infinite path*.

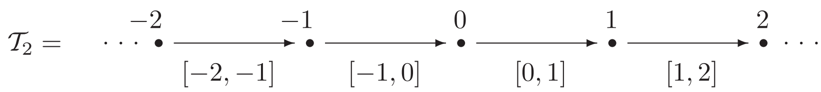


Figure 2.7: The oriented graph $\mathcal{T}_2$

Edge sequences and vertex sequences are used for infinite paths in the same way as for finite paths. Note that a sequence of edges is a function from some index set to an edge set, a sequence of vertices is a function from some index set to a vertex set. As for finite paths, the edge sequence of an infinite path determines the path entirely.

A finite path q will often be called a *segment*. This terminology is used most often, when the edge sequence $(a_1, \ldots, a_k)$ of q with length greater than zero appears in the sequence of a larger path p. If p has length n ($I = \{1, \ldots, n\}$), p is a ray ($I = \mathbb{N}$ or $I = -\mathbb{N}$) or a bi-infinite path ($I = \mathbb{Z}$) with edge sequence $p = \{e_i\}_{i \in I}$, then q is called a *segment of* p if and only if for some $m \in \mathbb{Z}$ holds $m + i \in I$ for all $i \in \{1, \ldots, k\}$ and then $e_{m+i} = a_i, \quad i = 1, \ldots, k$. A path q of length zero is a segment of a path p if the vertex $q(0)$ appears in the vertex sequence of p.

The oriented graph circ_n for $(n > 0)$ has n vertices $\{0, \ldots, n-1\}$ and n

positive edges $[0,1],\dots,[n-1,0]$, where $\mathrm{o}([i,i+1]) = i$ and $\mathrm{t}([i,i+1]) = i+1$ for all i ($i \bmod n$). There is a diagram of circ_n in Figure 2.8. An isomorphic image of circ_n is called a *circuit*. The number n is the *length* of a circuit. An isomorphic image of circ_1 is also called a *loop*.

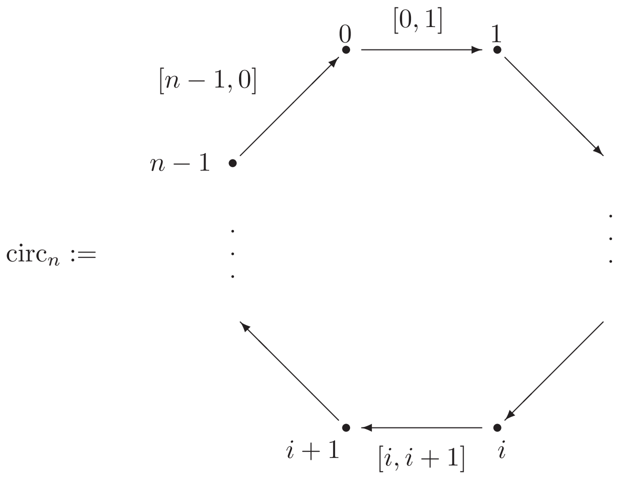


Figure 2.8: The oriented graph circ_n

A graph Γ is called *combinatorial* if it has no circuits of length ≤ 2. In this case, edges e of Γ can be written as ordered pairs of their borders

$$e = \big(\mathrm{o}(e), \mathrm{t}(e)\big). \tag{2.5}$$

This shall be proved. For all edges of Γ, one has $\mathrm{o}(e) \neq \mathrm{t}(e)$. Otherwise $[0,0] \mapsto e$ defines a loop in Γ. If there were two edges $a \neq b$ with the same borders $\mathrm{o}(a) = \mathrm{o}(b)$ and $\mathrm{t}(a) = \mathrm{t}(b)$, then their geometrical edges are disjoint except possibly $b = \overline{a}$ or $a = \overline{b}$. These cases are impossible because they imply $\mathrm{o}(a) = \mathrm{t}(\overline{a}) = \mathrm{t}(b) = \mathrm{t}(a)$. Then the geometric edges $\{a, \overline{a}\}$ and $\{b, \overline{b}\}$ are disjoint and the graph Γ has a circuit of length two. This contradicts the assumption that Γ was combinatorial.

Paths in a combinatorial graph can be written through their vertex sequence. In equation (2.3) paths have been established as sequences of edges. Edges can be written as ordered pairs of vertices by equation (2.5). Thus every path p in a combinatorial graph Γ can be defined by a sequence $(x_0,\dots,x_n)$ of vertices where each pair (x_i, x_{i+1}) is an edge.

2.4 Lemma. *A map* $F : \mathrm{V}\Gamma \mapsto \mathrm{V}\Gamma'$ *from the vertex set of a graph* Γ *to the vertex set of a combinatorial graph* Γ' *extends to a unique morphism* $\widetilde{F} : \Gamma \to \Gamma'$ *if and only if* F *maps adjacent vertices to adjacent vertices. In this case* $\widetilde{F}|_{\mathrm{V}\Gamma} = F|_{\mathrm{V}\Gamma}$. *For edges* $e \in \mathrm{E}\Gamma$ *one has* $\widetilde{F}(e) = \Big(F\big(\mathrm{o}(e)\big), F\big(\mathrm{t}(e)\big)\Big)$.

Proof. [Wei04] □

2.5 Lemma. *All isometries* $h \in \mathrm{Is}\,(\Gamma)$ *of a graph* Γ *are injective if and only if* Γ *is combinatorial.*

Proof. [Wei04] □

2.6 Definition (Trees)**.** A *tree* is a non-empty connected graph without circuits.

By definition, a tree is a combinatorial graph, thus paths can be written as vertex sequences. For an important class of paths in a tree, there is a more useful way of writing.

2.7 Definition (Geodesics)**.** A path (x, y, z) written as a vertex sequence is called a *reversal* if and only if $x = z$. A path g in a tree is called a *geodesic*, if and only if it has no reversals (as segments).

This definition applies to finite paths as well as to infinite paths. The condition says that g is a geodesic if and only if $g(i) \neq g(i+2)$ for all suitable i (such that $i, i+2 \in I$) where $I = \{0, \dots, n\}$, $I = \mathbb{N}_0$, $I = -\mathbb{N}_0$ or $I = \mathbb{Z}$ respectively, in the case that g is a finite path, a right ray, a left ray or a bi-infinite path respectively.

2.8 Proposition. *Two vertices in a tree are joined by a unique geodesic. This geodesic is an injective path.*

Proof. There is a very concise proof in [Ser80]. □

The unique geodesic joining a vertex x to a vertex y in a tree is denoted by

$$[x, y]. \tag{2.6}$$

2.9 Lemma. *Suppose $\mathcal{T}$ is a tree, $x, z \in \mathrm{V}(\mathcal{T})$ and p is a path from x to z. Then* $\mathrm{len}(p) = \mathrm{d}(x, z)$ *if and only if* $p = [x, z]$.

Proof. If $\mathrm{len}(p) = \mathrm{d}(x, z)$ then p is a geodesic, otherwise there is a path from x to z which has a length shorter than $\mathrm{len}(p) = \mathrm{d}(x, z)$. By uniqueness follows $p = [x, z]$. Conversely, by the definition of the distance d, there is a path q from x to z of length $\mathrm{d}(x, z)$, which is a geodesic by the above paragraph. By uniqueness or geodesics follows $p = [x, z] = q$, hence $\mathrm{len}(p) = \mathrm{len}(q) = \mathrm{d}(x, z)$. □

Chapter 3

Lines

Lines as geometrical objects of a tree $\mathcal{T}$ are introduced.

Geodesic Segments

A *geodesic segment* is a finite geodesic, i.e. a finite path $g : \text{path}_n \to \mathcal{T}$ of length n that has a vertex sequence $g(0), \ldots, g(n)$ without reversals. Justified through Proposition 2.8 the unique geodesic g joining two vertices x and y is denoted by

$$g = \big[x, y\big]. \tag{3.1a}$$

As usually for paths, the vertices of $g = [x, y]$ are written as $[x, y](i)$ for $0 \leq i \leq \text{len}([x, y])$. We say that a vertex x *lies* on a geodesic segment g (of length n), if $x \in \{g(0), \ldots, g(n)\}$ and write this as

$$x \in g. \tag{3.1b}$$

The *composition* of two geodesic segments p, q of lengths $n \geq 0$ and $m \geq 0$ respectively that satisfy $q(0) = p(n)$ is defined as the path

$$pq := \begin{cases} q & \text{if} \quad n = 0, \\ p & \text{if} \quad m = 0, \\ p(0), \ldots, p(n), q(1), \ldots, q(m) & \text{otherwise.} \end{cases}$$

Directly from Proposition 2.8 and Lemma 2.9, several equivalences are derived [Wei04]. Suppose three vertices $x, y, z \in \mathrm{V}(\mathcal{T})$ are given. Equivalent are

$$\begin{aligned}
&[x,y][y,z] = [x,z], \\
&[x,y][y,z] \quad \text{is a geodesic}, \\
&y \in [x,z], \\
&\mathrm{d}(x,y) + \mathrm{d}(y,z) = \mathrm{d}(x,z).
\end{aligned} \tag{3.1c}$$

If $h \in \mathrm{Is}(\mathcal{T})$ is an isometry and g is a geodesic segment, then the path $h(g) = h \circ g$ is also a geodesic segment, since isometries are injective. From uniqueness of geodesics follows for all $x, y \in \mathrm{V}(\mathcal{T})$ and all isometries $h \in \mathrm{Is}(\mathcal{T})$

$$h\big([x,y]\big) = \big[h(x), h(y)\big]. \tag{3.1d}$$

Geodesic Right-Rays

A *geodesic right-ray* is an infinite path $r : \mathrm{path}_\infty \to \mathcal{T}$ that has a vertex sequence $r = (r(0), r(1), r(2), \ldots)$ without reversals, i.e where $r(i) \neq r(i+2)$ for all $i \in \mathbb{N}_0$. The space of geodesic right-rays is denoted by $\mathcal{R}_\infty$.

The *relative boundary* of a tree $\mathcal{T}$ with respect to a given vertex $x \in \mathrm{V}\mathcal{T}$ is the set of geodesic right-rays

$$\mathcal{T}_x(\infty) = \big\{ r \in \mathcal{R}_\infty : \quad r(0) = x \big\}$$

that have *origin* $r(0) = x$.

3.1 Definition (Ideal Boundary of a Tree)**.** The *ideal boundary of a tree* is defined as the classes of an equivalence relation:

$$\mathcal{T}(\infty) := \mathcal{R}_\infty / \sim .$$

Two geodesics right-rays r and s are equivalent, if and only if they have the *infinite intersection property*, which requires constants $k_1, k_2 \in \mathbb{N}_0$ such that $r(k_1 + l) = s(k_2 + l)$ holds for all $l \in \mathbb{N}_0$.

The projection $\omega : \mathcal{R}_\infty \to \mathcal{T}(\infty)$ is called the *future*. For each fixed vertex $x \in \mathcal{T}$ the restriction $\omega|_{\mathcal{T}_x(\infty)}$ is a bijection from the relative boundary $\mathcal{T}_x(\infty)$ of x to the ideal boundary $\mathcal{T}(\infty)$.

$$\omega|_{\mathcal{T}_x(\infty)} : \begin{array}{ccl} \mathcal{T}_x(\infty) & \longrightarrow & \mathcal{T}(\infty) \\ r & \longmapsto & [r] \in \mathcal{T}(\infty) = \mathcal{R}_\infty/\sim . \end{array}$$

A proof can be found in [Wei04]. The identification will be used to write geodesic right-rays in terms of a vertex and a boundary point. For each vertex $x \in \mathcal{T}$ and each boundary point $\xi \in \mathcal{T}(\infty)$,

$$r = \big[x, \xi\big) \tag{3.2a}$$

denotes the the unique geodesic right-ray r with *origin* $r(0) = x$ and future $\omega(r) = \xi$. The ray r is called the ray *from x to ξ*. As usually, vertices of $r = [x, \xi)$ are written as $[x, \xi)(i)$ for $i \in \mathbb{N}_0$.

A vertex $x \in \mathrm{V}(\mathcal{T})$ is said to *lie* on a right-ray r if $x \in \big\{r(0), r(1), \dots\big\}$, which will be denoted by

$$x \in r. \tag{3.2b}$$

The *composition* gr of a geodesic segment g of length $n \geq 0$ and a geodesic right-ray r that satisfy $r(0) = g(n)$ is defined as the path

$$gr := \begin{cases} r & \text{if} \quad n = 0, \\ g(0), \dots, g(n), r(1), r(2), \dots & \text{otherwise.} \end{cases}$$

For all $x, y \in \mathrm{V}(\mathcal{T})$ and all boundary points $\xi \in \mathcal{T}(\infty)$ hold the following equivalences

$$\begin{aligned} &[x, y][y, \xi) = [x, \xi), \\ &[x, y][y, \xi) \quad \text{is a geodesic right-ray}, \\ &y \in [x, \xi). \end{aligned} \tag{3.2c}$$

Proof of (3.2c). The downward implications are trivial. For an implication from the third to the first statement we put $(r_0, r_1, r_2, \dots) := [x, \xi)$ for the vertex sequence of $[x, \xi)$. If $y \in [x, \xi)$ then $y = r_i$ for some $i \in \mathbb{N}_0$. Thus $[x, y] = (r_0, \dots, r_i)$ and $[y, \xi) = (r_i, r_{i+1}, \dots)$. The composition $[x, y][y, \xi)$ has thus the same vertex sequence as $[x, \xi)$, hence the two rays are equal. □

It was shown in [Wei04] that isometries $h \in \mathrm{Is}\,(\mathcal{T})$ induce maps $\mathcal{R}_\infty \to \mathcal{R}_\infty$ and they preserve the equivalence classes of $\mathcal{R}_\infty$, so that they induce maps $h : \mathcal{T}(\infty) \to \mathcal{T}(\infty)$. Further it was shown that they commute with the future map, i.e. $\omega\big(h(r)\big) = h\big(\omega(r)\big)$ for all isometries $h \in \mathrm{Is}\,(\mathcal{T})$ and all geodesic right-rays r. It follows then $h\Big(\big[r(0), \omega(r)\big)\Big) = h(r) = \Big[h(r)(0), \omega\big(h(r)\big)\Big) = \Big[h\big(r(0)\big), h\big(\omega(r)\big)\Big)$. This can be expressed by

$$h\big[x, \xi\big) = \big[h(x), h(\xi)\big) \tag{3.2d}$$

for all isometries $h \in \mathrm{Is}(\mathcal{T})$, all $x \in \mathrm{V}(\mathcal{T})$ and all $\xi \in \mathcal{T}(\infty)$.

Geodesic Left-Rays

A *geodesic left-ray* is an infinite path $l : \mathrm{path}_{-\infty} \to \mathcal{T}$ that has a vertex sequence $l = (\dots, l(-2), l(-1), l(0))$ without reversals, i.e where $l(i-2) \neq l(i)$ for all $i \in -\mathbb{N}_0$. The space of geodesics left-rays is denoted by $\mathcal{L}_\infty$.

Note that the space of geodesic left rays $\mathcal{L}_\infty$ is in a one-to-one correspondence with the space or geodesic right-rays. If r is a right-ray, then l defined by $l(i) := r(-i)$ for all $i \in -\mathbb{N}_0$ defines a geodesic left-ray and vice versa. Thus the properties of geodesic right-rays carry over to corresponding properties for geodesic left rays. The concepts need only be translated under the correspondence between $\mathcal{R}_\infty$ and $\mathcal{L}_\infty$.

The (anti)-projection $\alpha : \mathcal{L}_\infty \to \mathcal{T}(\infty)$ defined for geodesic left-rays l as $\alpha(l) := \omega\big(l(0), l(-1), l(-2), \dots\big)$ is called the *past*. For each vertex $x \in \mathcal{T}$ and each boundary point $\eta \in \mathcal{T}(\infty)$,

$$l = \big(\eta, x\big] \tag{3.3a}$$

denotes the the unique geodesic left-ray l with *terminus* $l(0) = x$ that has *past* $\alpha(l) = \eta$. The ray l is called the ray *from* η *to* x. As usually, vertices of $l = (\eta, x]$ are written as $(\eta, x](i)$ for $i \in -\mathbb{N}_0$. However we will most often write $[x, \eta)(i)$ for the vertex $(\eta, x](-i)$ for $i \in \mathbb{N}_0$.

A vertex $x \in \mathrm{V}(\mathcal{T})$ is said to *lie* on a left-ray l if $x \in \big\{l(0), l(-1), \dots\big\}$, which will be denoted by

$$x \in l. \tag{3.3b}$$

The *composition* lg of a geodesic left-ray l and a geodesic segment g of length $n \geq 0$ that satisfy $g(0) = l(0)$ is defined as the path

$$lg := \begin{cases} l & \text{if} \quad n = 0, \\ \dots, l(-1), l(0), g(1), \dots, g(n) & \text{otherwise.} \end{cases}$$

For all $x, y \in \mathrm{V}(\mathcal{T})$ and all boundary points $\eta \in \mathcal{T}(\infty)$ hold the following equivalences.

$$\begin{aligned}&(\eta, x][x, y] = (\eta, y],\\ &(\eta, x][x, y] \quad \text{is a geodesic left-ray,}\\ &x \in (\eta, y].\end{aligned} \tag{3.3c}$$

For all isometries $h \in \mathrm{Is}\,(\mathcal{T})$, all $x \in \mathrm{V}(\mathcal{T})$ and all $\eta \in \mathcal{T}(\infty)$ holds

$$h\big(\eta, x\big] = \big(h(\eta), h(x)\big]. \tag{3.3d}$$

Bi-infinite Lines

A *bi-infinite geodesic* is a bi-infinite path $g : \mathcal{T}_2 \to \mathcal{T}$ that has a vertex sequence $\ldots, g(-1), g(0), g(1), \ldots$ without reversals, i.e. $g(i) \neq g(i+2)$ for all $i \in \mathbb{Z}$. The set of all bi-infinite geodesics is denoted by $\mathcal{G}$ and is called the *geodesic space* (of $\mathcal{T}$). There is an invertible *shift operator* $\mathrm{L} : \mathcal{G} \to \mathcal{G}$ defined for geodesics g by

$$\big(\mathrm{L}(g)\big)(i) := g(i+1), \quad i \in \mathbb{Z}.$$

The operator L acts on $\mathcal{G}$ because for $g \in \mathcal{G}$ all shifted geodesics $\mathrm{L}^n(g)$ also pertain to $\mathcal{G}$ for all $n \in \mathbb{Z}$. More generally, the shift operator L acts on the space $\mathcal{P}$ of all bi-infinite paths (not necessarily geodesics) in $\mathcal{T}$. The *velocity* $\mathcal{V}(g)$ of a geodesic $g \in \mathcal{G}$ is defined by a function

$$\mathcal{V} : \begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{T}(\infty) \times \mathcal{T}(\infty) \\ g & \longmapsto & \big(\,\alpha(g),\, \omega(g)\,\big). \end{array} \tag{3.4}$$

It consists of two boundary points; $\alpha(g) := \alpha\big(\ldots, g(-1), g(0)\big)$ is the past of the denoted left-ray and $\omega(g) := \omega\big(g(0), g(1), \ldots\big)$ is the future of the given right-ray. We call $\alpha(g)$ the *past* of g and $\omega(g)$ the *future* of g. With the diagonal $\Delta := \big\{(\gamma, \gamma) : \gamma \in \mathcal{T}(\infty)\big\}$ the *velocity space* of $\mathcal{T}$ is defined as

$$\mathcal{V}_{\mathcal{T}} := \big(\mathcal{T}(\infty) \times \mathcal{T}(\infty)\big) - \Delta.$$

Theorem 1. *The velocity $\mathcal{V}$ is a surjective map $\mathcal{G} \to \mathcal{V}_{\mathcal{T}}$. Each two geodesics $g, h \in \mathcal{G}$ that have the same velocity are linked by the shift operator $h = \mathrm{L}^n(g)$ for a unique $n \in \mathbb{Z}$. There is a bijection $\mathcal{G} = \mathcal{V}_{\mathcal{T}} \times \mathbb{Z}$. The velocity $\mathcal{V}$ induces a bijection $\widetilde{\mathcal{V}} : \mathcal{G}/_{\mathbb{Z}} \longrightarrow \mathcal{V}_{\mathcal{T}}$.*

Proof. Since every geodesic $g \in \mathcal{G}$ is injective one has $\omega(g) \neq \alpha(g)$. Otherwise the rays $\big[g(0), g(1), \dots\big)$ and $\big[g(0), g(-1), \dots\big)$ had infinite intersection.

For surjectivity of $\mathcal{V}$ we may fix a vertex $x \in \mathrm{V}(\mathcal{T})$ and two distinct boundary points $\eta \neq \xi$. The rays $[x, \eta)$ and $[x, \xi)$ have no infinite intersection, thus there is a natural number $n \in \mathbb{N}_0$ such that $\big[x, \eta\big)(n) = \big[x, \xi\big)(n)$ and $\big[x, \eta\big)(n+1) \neq \big[x, \xi\big)(n+1)$. Then

$$g(i) := \begin{cases} \big[x, \eta\big)(n-i) & \text{for} \quad i \leq 0 \\ \big[x, \xi\big)(n+i) & \text{for} \quad i > 0 \end{cases}$$

defines a geodesic $g \in \mathcal{G}$ with the required property $\mathcal{V}(g) = (\eta, \xi)$.

Let two geodesics $g, h \in \mathcal{G}$ have the same velocity $\mathcal{V}(g) = \mathcal{V}(h)$. Since g and h have the same past, there are integers $l_1, l_2 \leq 0$ with $g(l_1) = h(l_2)$. The geodesics have the same future, thus there are integers $r_1, r_2 \geq 0$ with $g(r_1) = h(r_2)$. By uniqueness of geodesics (Prop. 2.8) one has $[g(l_1), g(r_1)] = [h(l_2), h(r_2)]$. The lengths of these paths are equal, which permits to put $k := l_2 - l_1 = r_2 - r_1$. A straight calculation (with a selection of three cases) gives then $g(i) = h(i+k)$ for all $i \in \mathbb{Z}$. Thus $g = \mathrm{L}^k(h)$. Since geodesics are injective paths, every vertex of the vertex sequence of g appears only once, so k is unique.

For each velocity $v \in \mathcal{V}$ we choose a geodesic $\mathbf{g}_v$ with $\mathcal{V}(\mathbf{g}_v) = v$. Then a bijection $\mathcal{G} \to \mathcal{V}_\mathcal{T} \times \mathbb{Z}$ is obtained by $g \mapsto \big(\mathcal{V}(g), k\big)$ with the unique $k \in \mathbb{Z}$ such that $\mathbf{g}_v = \mathrm{L}^k(g)$. This map is injective by the second statement of the theorem. Next we shows that $\mathcal{V}\big(L^n(g)\big) = \mathcal{V}(g)$ for all $n \in \mathbb{Z}$. Then it follows from the first assertion of the theorem, that the map is surjective. One has $\omega\big(L^n(g)\big) = \omega\big(g(n), g(n+1), \dots\big)$ which equals $\omega(g) = \omega\big(g(0), g(1), \dots\big)$ by infinite intersection. Similarly $\alpha\big(L^n(g)\big) = \alpha(g)$ for all $g \in \mathcal{G}$ and all $k \in \mathbb{Z}$.

The invariance of $\mathcal{V}$ under the shift operator L shows that $\mathcal{V}$ is constant on the orbits of the $\mathbb{Z}$-action of L on $\mathcal{G}$ so it induces a map $\widetilde{\mathcal{V}} : \mathcal{G}/_\mathbb{Z} \to \mathcal{V}_\mathcal{T}$. The first statement of this proposition proves that this map is surjective, the second statement proves that it is injective. □

The elements of the velocity space $\mathcal{V}_\mathcal{T}$ are called *bi-infinite lines*. A bi-infinite line $l \in \mathcal{V}_\mathcal{T}$ is written as

$$l = \big(\eta, \xi\big). \tag{3.5a}$$

Any geodesic $g \in \mathcal{G}$ that has velocity $\mathcal{V}(g) = l$ is called a geodesic *representing* the line l.

We say a vertex $x \in \mathrm{V}(\mathcal{T})$ *lies* on a bi-infinite line l if and only if x pertains to the vertex set of some (hence by Theorem 1 every) geodesic g with velocity l. This will be denoted as

$$x \in l. \tag{3.5b}$$

When a geodesic left-ray l and a geodesic right-ray satisfy $r(0) = l(0)$ then the *composition* lr is defined: a bi-infinite path p is given as $p(i) := l(i)$ for $i \leq 0$ and $p(i) = r(i)$ otherwise. The composition is then defined as the projection of p to the orbit of p under the action of the shift operator L.

$$lr := p\mathbb{Z} \in \mathcal{P}/_{\mathbb{Z}}.$$

For all boundary points $\eta \neq \xi$ and all vertices $x \in \mathrm{V}(\mathcal{T})$ hold the following equivalences.

$$\begin{aligned} &(\eta, x][x, \xi) = (\eta, \xi), \\ &(\eta, x][x, \xi) \quad \text{is a bi-infinite line}, \\ &x \in (\eta, \xi). \end{aligned} \tag{3.5c}$$

Proof of (3.5c). Recall from Eqn. (3.5a) that the symbol (η, ξ) is reserved for bi-infinite lines (and implies $\eta \neq \xi$). The statement $(\eta, x][x, \xi) = (\eta, \xi)$ says that the composition of $(\eta, x]$ with $[x, \xi)$ equals the line $(\eta, \xi) \in \mathcal{G}/_{\mathbb{Z}} \subset \mathcal{P}/_{\mathbb{Z}}$. In particular, the composition is a bi-infinite line.

For the next step, we recapitulate the definition of the composition. It is given by the orbit under the shift operator L of the bi-infinite path p defined as $p(i) := [x, \eta)(i)$ for $i \leq 0$, and as $p(i) := [x, \xi)(i)$ otherwise. Note that the action of L on $\mathcal{P}$ preserves the geodesic space, i.e. $\mathrm{L}(\mathcal{G}) \subset \mathcal{G}$. So, since the composition is a line, p is a bi-infinite geodesic. Obviously p represents the line (η, ξ). Since x is a vertex of p this shows $x \in (\eta, \xi)$.

If $x \in (\eta, \xi)$ then there is a bi-infinite geodesic g with past $\alpha(g) = \eta$, with future $\omega(g) = \xi$ and such that $x = g(k)$ for some $k \in \mathbb{Z}$. By Theorem 1 we can assume that $k = 0$. It follows from Eqn. (3.3a) and Eqn. (3.2a) by uniqueness of geodesic rays that $(\eta, x] = \big(\ldots, g(-1), g(0)\big]$ and $[x, \xi) = \big[g(0), g(1), \ldots\big)$. So the path p that appears in the definition of the composition of the rays $(\eta, x]$ and $[x, \xi)$ equals g. Since g represents the line (η, ξ), the proof is finished. $\square$

For all isometries $h \in \mathrm{Is}(\mathcal{T})$ and all boundary points $\eta \neq \xi$ holds

$$h(\eta, \xi) = \big(h(\eta), h(\xi)\big). \tag{3.5d}$$

Proof of (3.5d). We have to show that for every geodesic g representing (η, ξ) hold $h\alpha(g) = h\eta$ and $h\omega(g) = \omega(\eta)$. One has $h\alpha(g) = h\alpha(\eta, g(0)] = h\eta$ as well as $h\omega(g) = h\omega[g(0), \xi) = h\xi$. □

Lines in a Tree

The distinct types of lines introduced so far have very similar behavior. This allows a unification. For symbols $\mathbf{x}, \mathbf{y} \in \mathcal{T} \cup \mathcal{T}(\infty)$ such that $\mathbf{y} \neq \mathbf{x}$ whenever $\mathbf{x}, \mathbf{y} \in \mathcal{T}(\infty)$ we use the notation

$$\langle \mathbf{x}, \mathbf{y} \rangle \tag{3.6a}$$

to refer with the ordered symbols $\mathbf{x}, \mathbf{y}$ in angle brackets '$\langle$' and '$\rangle$' to the corresponding definition (3.1a), (3.2a), (3.3a) or (3.5a). This means that $\langle \mathbf{x}, \mathbf{y} \rangle$ stands for the

geodesic segment	$[\mathbf{x}, \mathbf{y}]$	if $\mathbf{x}, \mathbf{y} \in \mathcal{T}$,
geodesic right-ray	$[\mathbf{x}, \mathbf{y})$	if $\mathbf{x} \in \mathcal{T}$ and $\mathbf{y} \in \mathcal{T}(\infty)$,
geodesic left-ray	$(\mathbf{x}, \mathbf{y}]$	if $\mathbf{x} \in \mathcal{T}(\infty)$ and $\mathbf{y} \in \mathcal{T}$,
bi-infinite line	$(\mathbf{x}, \mathbf{y})$	if $\mathbf{x}, \mathbf{y} \in \mathcal{T}(\infty)$.

In all cases the object $\langle \mathbf{x}, \mathbf{y} \rangle$ is called a *line* in $\mathcal{T}$.

We simply copy the remaining concepts from the different types of lines. A vertex z is said to *lie* on a line $\langle \mathbf{x}, \mathbf{y} \rangle$ if

$$z \in \langle \mathbf{x}, \mathbf{y} \rangle. \tag{3.6b}$$

For all symbols $\mathbf{x}, \mathbf{y} \in \mathrm{V}\mathcal{T} \cup \mathcal{T}(\infty)$ (maybe equal boundary points) and all vertices $z \in \mathrm{V}(\mathcal{T})$ there is a *composition*

$$\langle \mathbf{x}, z][z, \mathbf{y} \rangle$$

such that the following statements are equivalent for all $\mathbf{x}, \mathbf{y} \in \mathcal{T} \cup \mathcal{T}(\infty)$ and all $z \in \mathrm{V}(\mathcal{T})$.

$$\begin{aligned} &\langle \mathbf{x}, z][z, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle, \\ &\langle \mathbf{x}, z][z, \mathbf{y} \rangle \quad \text{is a line,} \\ &z \in \langle \mathbf{x}, \mathbf{y} \rangle. \end{aligned} \tag{3.6c}$$

For all isometries $h \in \mathrm{Is}(\mathcal{T})$ and all lines $\langle \mathbf{x}, \mathbf{y} \rangle$ holds

$$h\big(\langle \mathbf{x}, \mathbf{y} \rangle\big) = \big\langle h(\mathbf{x}), h(\mathbf{y}) \big\rangle. \tag{3.6d}$$

When we work in a situation where it is clear whether two symbols $\mathbf{x}, \mathbf{y} \in \mathcal{T} \cup \mathcal{T}(\infty)$ are vertices or boundary points, we will use the old notation with squared brackets '[' and ']' and parenthesis '(' and ')'. In particular, we will often keep writing $[\mathbf{x}, \mathbf{y}]$ for geodesic segments, $[\mathbf{x}, \mathbf{y})$ for right-rays, $(\mathbf{x}, \mathbf{y}]$ for left-rays and $(\mathbf{x}, \mathbf{y})$ for bi-infinite lines. The expression $[\mathbf{x}, \mathbf{y}\rangle$ is used, when $\mathbf{x} \in \mathcal{T}$ and there is no specification about $\mathbf{y} \in \mathcal{T} \cup \mathcal{T}(\infty)$. Similarly we will use $\langle \mathbf{x}, \mathbf{y}]$.

The *vertex set* of a line $\langle \mathbf{x}, \mathbf{y} \rangle$ is defined as the set of vertices that lie on this line. We will always state explicitly when a relation for vertex sets of lines is considered, because a line has more information as its vertex set.

3.2 Proposition. *For the vertex sets of lines hold the following relations (not for the lines themselves!). Let* $z \in \mathrm{V}(\mathcal{T})$ *and* $\langle \mathbf{x}, \mathbf{y} \rangle$ *be a line. Then*

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \langle \mathbf{y}, \mathbf{x} \rangle, \\ z \in \langle \mathbf{x}, \mathbf{y} \rangle \quad &\text{if and only if} \quad [z, \mathbf{y}\rangle \subset \langle \mathbf{x}, \mathbf{y} \rangle, \\ \text{if} \quad z \in \langle \mathbf{x}, \mathbf{y} \rangle \quad &\text{then} \quad \langle \mathbf{x}, z] \cup [z, \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y} \rangle, \\ \text{if} \quad z \in \langle \mathbf{x}, \mathbf{y} \rangle \quad &\text{then} \quad \langle \mathbf{x}, z] \cap [z, \mathbf{y}\rangle = \{z\} \end{aligned}$$

Proof. The first assertion is trivial. The leftward arrow of the second assertion is obvious. If $z \in \langle \mathbf{x}, \mathbf{y} \rangle$ then by Eqn. 3.6c holds $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, z][z, \mathbf{y}\rangle$. So the result $[z, \mathbf{y}\rangle \subset \langle \mathbf{x}, \mathbf{y} \rangle$ follows directly from the definition of the composition. From there follows also the third assertion. The last statement is a consequence of injectivity of geodesics. □

3.3 Corollary. *If* $\langle \mathbf{x}, \mathbf{y} \rangle$ *is a line and* $c, a, b \in \langle \mathbf{x}, \mathbf{y} \rangle$ *then* $c \notin [a, b]$ *implies* $c \in [a, \mathbf{x}\rangle \cap [b, \mathbf{x}\rangle$ *or* $c \in [a, \mathbf{y}\rangle \cap [b, \mathbf{y}\rangle$.

Proof. We use Proposition 3.2. Since $a \in \langle \mathbf{x}, \mathbf{y} \rangle$, one has $\langle \mathbf{x}, \mathbf{y} \rangle = [a, \mathbf{x}\rangle \cup [a, \mathbf{y}\rangle$. Since $b \in \langle \mathbf{x}, \mathbf{y} \rangle$, it follows then that $b \in [a, \mathbf{x}\rangle$ or $b \in [a, \mathbf{y}\rangle$. Thus, after possibly relabeling $a \leftrightarrow b$ one can assume that $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, a][a, b][b, \mathbf{y}\rangle$ is a line. The assumption $c \notin [a, b]$ allows then two cases, $c \in \langle \mathbf{x}, a]$ or $c \in [b, \mathbf{y}\rangle$. The proof is finished by observing that $\langle \mathbf{x}, a] \subset \langle \mathbf{x}, a][a, b] = \langle \mathbf{x}, b]$ and $[b, \mathbf{y}\rangle \subset [a, b][b, \mathbf{y}\rangle = [a, \mathbf{y}\rangle$. $\square$

Chapter 4

Triangles

A *triangle* in a tree $\mathcal{T}$ is defined as an ordered tuple $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ consisting of three *points* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{T} \cup \mathcal{T}(\infty)$ such that no two of the points are coinciding boundary points. The lines $\langle \mathbf{x}, \mathbf{y} \rangle$, $\langle \mathbf{y}, \mathbf{z} \rangle$ and $\langle \mathbf{z}, \mathbf{x} \rangle$ are called the *sides* of the triangle $(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

4.1 Lemma (Bifurcation Lemma)**.** *We assume that $\Delta = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a triangle in a tree $\mathcal{T}$. The intersection $\langle \mathbf{x}, \mathbf{y} \rangle \cap \langle \mathbf{y}, \mathbf{z} \rangle \cap \langle \mathbf{z}, \mathbf{x} \rangle$ of the three sides of Δ consists of a single vertex $\mathfrak{b}(\Delta) = \mathfrak{b}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathcal{T}$.*

Proof. For existence of $\mathfrak{b}(\Delta)$, we look in a first step for a vertex p, which lies on the two lines $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\langle \mathbf{z}, \mathbf{y} \rangle$. Such a vertex is always found. If $\mathbf{y}$ is a vertex, then the choice $p = \mathbf{y}$ is sufficient. If $\mathbf{y}$ is a boundary point, then $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\langle \mathbf{z}, \mathbf{y} \rangle$ have infinite intersection. Any choice of p in this intersection is valid. Commencing with p, a vertex in the intersection of the three sides of Δ can be constructed inductively: If $\langle \mathbf{x}, p][p, \mathbf{z} \rangle$ is a line, then p lies on the side $\langle \mathbf{x}, \mathbf{z} \rangle$, too. Otherwise $[p, \mathbf{x}\rangle(1) = [p, \mathbf{z}\rangle(1)$ and this vertex lies on the lines $\langle \mathbf{y}, \mathbf{x} \rangle$ and $\langle \mathbf{y}, \mathbf{z} \rangle$ since p does. Inductively we assume that $[p, \mathbf{x}\rangle(n) = [p, \mathbf{z}\rangle(n)$ and that this vertex lies on the lines $\langle \mathbf{y}, \mathbf{x} \rangle$ and $\langle \mathbf{y}, \mathbf{z} \rangle$. If $\langle \mathbf{x}, [p, \mathbf{x}\rangle(n)][[p, \mathbf{z}\rangle(n), \mathbf{z} \rangle$ is a line, then $[p, \mathbf{z}\rangle(n)$ lies on the side $\langle \mathbf{x}, \mathbf{z} \rangle$ and the proof of existence is finished. If the composition if not a line, then

$$[p, \mathbf{x}\rangle(n+1) = \big[[p, \mathbf{x}\rangle(n), \mathbf{x}\big\rangle(1) = \big[[p, \mathbf{z}\rangle(n), \mathbf{z}\big\rangle(1) = \big[p, \mathbf{z}\big\rangle(n+1)$$

and this vertex lies on $\langle \mathbf{y}, \mathbf{x} \rangle$ and on $\langle \mathbf{y}, \mathbf{z} \rangle$ as the vertex p does.

In the case that for $n = 0, 1, 2, \ldots$ none of the vertices $[p, \mathbf{z}\rangle(n)$ lies on $\langle \mathbf{x}, \mathbf{z}\rangle$ then there are identities $[p, \mathbf{x}\rangle(n+1) = [p, \mathbf{z}\rangle(n+1)$ for $n = 0, 1, 2, \ldots$. These identities show that the geodesics $[p, \mathbf{x}\rangle$ and $[p, \mathbf{z}\rangle$ have infinite lengths and infinite intersection, so $\mathbf{x}, \mathbf{z} \in \mathcal{T}(\infty)$ and $\mathbf{x} = \mathbf{z}$. This case was excluded in the definition of a triangle.

To establish uniqueness of a vertex that lies on the three sides of a triangle Δ, we assume s and t were two candidates. It can be assumed that $\langle \mathbf{x}, \mathbf{z}\rangle = \langle \mathbf{x}, s][s, t][t, \mathbf{z}\rangle$. Then both equations $\langle \mathbf{y}, \mathbf{x}\rangle = \langle \mathbf{y}, t][t, \mathbf{x}\rangle = \langle \mathbf{y}, t][t, s][s, \mathbf{x}\rangle$ and $\langle \mathbf{y}, \mathbf{z}\rangle = \langle \mathbf{y}, s][s, \mathbf{z}\rangle = \langle \mathbf{y}, s][s, t][t, \mathbf{z}\rangle$ hold. By uniqueness of geodesics, we can therefore substitute $\langle \mathbf{y}, s] = \langle \mathbf{y}, t][t, s] = \langle \mathbf{y}, s][s, t][t, s]$. The path $[s, t][t, s]$ as a geodesic is injective, hence must have length zero. This shows $s = t$. □

4.2 Definition (Bifurcations)**.** Given a triangle $\Delta = \big(\mathbf{x}, \mathbf{y}, \mathbf{z}\big)$, the vertex $\mathfrak{b}\big(\Delta\big) = \mathfrak{b}\big(\mathbf{x}, \mathbf{y}, \mathbf{z}\big)$ identified in the Bifurcation Lemma, is called the *bifurcation of the triangle* Δ or the *bifurcation* of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

4.3 Note. For every triangle $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and each permutation $\sigma \in S_3$ holds $\mathfrak{b}\big(\sigma(\mathbf{x}), \sigma(\mathbf{y}), \sigma(\mathbf{z})\big) = \mathfrak{b}(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

4.4 Example (Bifurcations)**.**

- For a fixed vertex $x \in \mathcal{T}$, a metric was defined [Wei04] on the ideal boundary of a tree by

$$\mathrm{d}_x(\eta, \xi) := \begin{cases} 0 & \text{if} \quad \eta = \xi \\ e^{-\mathrm{d}(x, \mathfrak{b}(x, \eta, \xi))} & \text{otherwise.} \end{cases}$$

  for boundary points $\eta, \xi \in \mathcal{T}(\infty)$.

- For a fixed boundary point $\gamma \in \mathcal{T}(\infty)$ a horocycle distance will be defined in Chapter 7 for vertices $x, y \in \mathcal{T}$ by

$$B_\gamma(x, y) := \mathrm{d}(y, \mathfrak{b}(x, y, \gamma)) - \mathrm{d}(x, \mathfrak{b}(x, y, \gamma)).$$

4.5 Definition (Sub-lateral triangles)**.** Two triangles Δ_S and Δ_T of a tree shall have sides S_1, S_2, S_3 and T_1, T_2, T_3 respectively. The triangle Δ_S is called *sub-lateral* to Δ_T, if $S_1 \subset T_1$, $S_2 \subset T_2$ and $S_3 \subset T_3$. Sometimes we will abbreviate that Δ_S is sub-lateral to Δ_T by writing $\Delta_S < \Delta_T$.

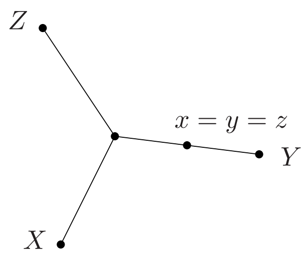


Figure 4.1: Each side of the triangle $\Delta_1 = (x, y, z)$ (consisting of a single vertex $x = y = z \in \mathcal{T}$) is included in some side of a triangle $\Delta_2 = (X, Y, Z)$ (for $X, Y, Z \in \mathcal{T}$). In fact, all sides of Δ_1 are included in $[X, Y]$ and in $[Y, Z]$. Since there is no side of Δ_1, that is included in $[Z, X]$, the triangle Δ_1 is not sub-lateral to Δ_2. Proposition 4.6 does not apply. Indeed, Δ_1 and Δ_2 have different bifurcations.

Triangles of a tree can be compared. The following Proposition displays a useful property of sub-lateral triangles. This feature fails under a weaker form of "inclusion" for triangles. See Figure 4.1 for an example.

4.6 Proposition (Sub-lateral triangles)**.**

- *If* Δ_S *and* Δ_T *are two triangles in a tree and* Δ_S *is sub-lateral to* Δ_T *then* $\mathfrak{b}(\Delta_S) = \mathfrak{b}(\Delta_T)$.

- *The bifurcation* $\mathfrak{b}(\Delta)$ *of a triangle* $\Delta = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ *is a projection in the sense that* $\mathfrak{b}\big(\mathfrak{b}(\Delta), \mathbf{y}, \mathbf{z}\big) = \mathfrak{b}(\Delta)$.

Proof. Bifurcations of triangles are unique and

$$\{\mathfrak{b}(\Delta_S)\} = S_1 \cap S_2 \cap S_3 \subset T_1 \cap T_2 \cap T_3 = \{\mathfrak{b}(\Delta_T)\}.$$

(The notation is borrowed from the definition of sub-lateral triangles.)

For the second statement, one has $\big[\mathfrak{b}(\Delta), \mathbf{y}\big\rangle \subset \langle \mathbf{x}, \mathbf{y}\rangle$ and $\big\langle \mathbf{z}, \mathfrak{b}(\Delta)\big] \subset \langle \mathbf{z}, \mathbf{x}\rangle$ by definition of $\mathfrak{b}(\Delta)$. This shows that $\big(\mathfrak{b}(\Delta), \mathbf{y}, \mathbf{z}\big) < (\mathbf{x}, \mathbf{y}, \mathbf{z})$. The first statement of this proposition implies now $\mathfrak{b}\big(\mathfrak{b}(\Delta), \mathbf{y}, \mathbf{z}\big) = \mathfrak{b}(\mathbf{x}, \mathbf{y}, \mathbf{z})$. □

4.7 Example (The bifurcation as a projection)**.** We consider a triangle $\Delta =$

$(\mathbf{x},\mathbf{y},\mathbf{z})$ in a tree.

$$\begin{aligned}
\mathfrak{b}(\Delta) &\overset{\text{Prop.4.6}}{=} \mathfrak{b}\big(\mathfrak{b}(\Delta),\mathbf{y},\mathbf{z}\big) \overset{\text{Prop.4.6}}{=} \mathfrak{b}\Big(\mathfrak{b}(\Delta),\mathfrak{b}\big(\mathfrak{b}(\Delta),\mathbf{y},\mathbf{z}\big),\mathbf{z}\Big)\\
&= \mathfrak{b}\big(\mathfrak{b}(\Delta),\mathfrak{b}(\Delta),\mathbf{z}\big) \overset{\text{Prop.4.6}}{=} \mathfrak{b}\Big(\mathfrak{b}(\Delta),\mathfrak{b}(\Delta),\mathfrak{b}\big(\mathfrak{b}(\Delta),\mathfrak{b}(\Delta),\mathbf{z}\big)\Big)\\
&= \mathfrak{b}\big(\mathfrak{b}(\Delta),\mathfrak{b}(\Delta),\mathfrak{b}(\Delta)\big).
\end{aligned}$$

For isometries h we define the image of a triangle $\Delta = (\mathbf{x},\mathbf{y},\mathbf{z})$ under h as the triangle

$$h(\Delta) := (h\mathbf{x}, h\mathbf{y}, h\mathbf{z}).$$

4.8 Proposition (Commutation of bifurcations)**.** *If Δ is a triangle in a tree and and h is an isometry then* $\mathfrak{b}\big(h(\Delta)\big) = h\big(\mathfrak{b}(\Delta)\big)$

Proof. Say $\Delta = (\mathbf{x},\mathbf{y},\mathbf{z})$. By definition $\mathfrak{b}(\Delta) \in \langle \mathbf{x},\mathbf{y}\rangle$. Thus $h\big(\mathfrak{b}(\Delta)\big) \in h\big(\langle \mathbf{x},\mathbf{y}\rangle\big)$. This line equals by Eqn. (3.6d) the line $\big\langle h(\mathbf{x}), h(\mathbf{y})\big\rangle$. Similarly one shows that $h\big(\mathfrak{b}(\Delta)\big) \in \big\langle h(\mathbf{y}), h(\mathbf{z})\big\rangle$ and $h\big(\mathfrak{b}(\Delta)\big) \in \big\langle h(\mathbf{z}), h(\mathbf{x})\big\rangle$. Then the Bifurcation Lemma shows that $h\big(\mathfrak{b}(\Delta)\big) = \mathfrak{b}\big(h(\Delta)\big)$. $\square$

Given a triangle $\Delta = (\mathbf{x},\mathbf{y},\mathbf{z})$, the set

$$\mathrm{V}(\Delta) := \langle \mathbf{x},\mathbf{y}\rangle \cup \langle \mathbf{y},\mathbf{z}\rangle \cup \langle \mathbf{z},\mathbf{x}\rangle$$

is called the *vertex set* of Δ.

4.9 Lemma (Partition of triangles)**.** *We assume that $\Delta = (\mathbf{x},\mathbf{y},\mathbf{z})$ is a triangle in a tree. Then*

(a) $\big[\mathfrak{b}(\Delta),\mathbf{x}\big\rangle = \langle \mathbf{y},\mathbf{x}\rangle \cap \langle \mathbf{z},\mathbf{x}\rangle$

(b) $\big[\mathfrak{b}(\Delta),\mathbf{x}\big\rangle \cap \big[\mathfrak{b}(\Delta),\mathbf{y}\big\rangle = \big\{\mathfrak{b}(\Delta)\big\}$

(c) $V(\Delta) \setminus \big\{\mathfrak{b}(\Delta)\big\} = \Big(\big[\mathfrak{b}(\Delta),\mathbf{x}\big\rangle \setminus \big\{\mathfrak{b}(\Delta)\big\}\Big) \sqcup \Big(\big[\mathfrak{b}(\Delta),\mathbf{y}\big\rangle \setminus \big\{\mathfrak{b}(\Delta)\big\}\Big) \sqcup \Big(\big[\mathfrak{b}(\Delta),\mathbf{z}\big\rangle \setminus \big\{\mathfrak{b}(\Delta)\big\}\Big).$

Proof. In (a), the inclusion from left to right comes from Prop. 3.2, since $\mathfrak{b}(\Delta)$ lies on both lines $\langle \mathbf{y},\mathbf{x}\rangle$ and $\langle \mathbf{z},\mathbf{x}\rangle$. Conversely, we assume that a vertex c lies on $\langle \mathbf{y},\mathbf{x}\rangle$ and on $\langle \mathbf{z},\mathbf{x}\rangle$ but $c \notin \big[\mathfrak{b}(\Delta),\mathbf{x}\big\rangle$ then $c \in \big\langle \mathbf{z},\mathfrak{b}(\Delta)\big]$ because $c \in \langle \mathbf{z},\mathbf{x}\rangle$. This shows that $c \in \langle \mathbf{z},\mathbf{y}\rangle$ and thus $c = \mathfrak{b}(\Delta)$, a contradiction.

The assertion (b) follows directly from Prop. 3.2 because $\mathfrak{b}(\Delta) \in \langle \mathbf{x},\mathbf{y}\rangle$.

To verify (c), note that $\mathrm{V}(\Delta) = \big[\mathfrak{b}(\Delta),\mathbf{x}\big\rangle \cup \big[\mathfrak{b}(\Delta),\mathbf{y}\big\rangle \cup \big[\mathfrak{b}(\Delta),\mathbf{z}\big\rangle$ because $\mathfrak{b}(\Delta)$ lies on all three sides of Δ. If the union is not disjoint, then a vertex c is in two of the involved sets, say $c \in \big[\mathfrak{b}(\Delta), \mathbf{x}\big\rangle \cap \big[\mathfrak{b}(\Delta), \mathbf{y}\big\rangle$. This implies by part (b) of this lemma that $c = \mathfrak{b}(\Delta)$. $\square$

The *distance of a vertex x to a line $l = \langle \mathbf{y}, \mathbf{z} \rangle$* is defined as

$$\mathrm{d}(x, l) := \min_{y \in l} \big\{ \mathrm{d}(x, y) \big\}.$$

4.10 Lemma (Distance to a Line)**.** *For each vertex x and each line $l = \langle \mathbf{y}, \mathbf{z} \rangle$ holds $\mathrm{d}\big(x, \mathfrak{b}(x, \mathbf{y}, \mathbf{z})\big) = \mathrm{d}(x, l)$. The bifurcation $\mathfrak{b}(x, \mathbf{y}, \mathbf{z})$ is the unique vertex of l in minimal distance to x.*

Proof. By well-ordering of the natural numbers, at least one vertex $z \in l$ is in distance $\mathrm{d}(x, z) = \mathrm{d}(x, l)$ to x. If $\big[z, x\big](1) = \big[z, \mathbf{y}\big\rangle(1)$ then $\big[z, x\big](1)$ is also a vertex of l with a distance to x smaller than $\mathrm{d}(x, z)$, a contradiction. Thus $[x, z][z, \mathbf{y}\rangle$ is a geodesic. By the same arguments is $z \in [x, \mathbf{z}\rangle$. This shows $z = \mathfrak{b}(x, \mathbf{y}, \mathbf{z})$. $\square$

Chapter 5

Quadrilaterals

A *quadrilateral* is an ordered quadruple $A = (1_A, 2_A, 3_A, 4_A)$ consisting of four *points* $1_A, 2_A, 3_A, 4_A \in \mathcal{T} \cup \mathcal{T}(\infty)$ such that no two of these points are coinciding boundary points. We emphasize the indices of points rather than the quadrilateral A, since they will become more important later on. We denote by $\mathcal{T}_Q$ the set of all quadrilaterals of a tree $\mathcal{T}$. A quadrilateral A has four bifurcations

$$\begin{aligned} \mathfrak{b}(1)_A &:= \mathfrak{b}(2_A, 3_A, 4_A), & \mathfrak{b}(2)_A &:= \mathfrak{b}(1_A, 3_A, 4_A), \\ \mathfrak{b}(3)_A &:= \mathfrak{b}(1_A, 2_A, 4_A) \quad \text{and} & \mathfrak{b}(4)_A &:= \mathfrak{b}(1_A, 2_A, 3_A). \end{aligned}$$

5.1 Mappings by Isometries and Permutations

For this section it is convenient to think of a quadrilateral A as a sequence $(i_A)_{i \in \{1,2,3,4\}}$ or likewise as a map $A : \{1, 2, 3, 4\} \mapsto \mathcal{T} \cup \mathcal{T}(\infty)$. For subsets $S \subset \{1, 2, 3, 4\}$ we use the notation $A(S) := A|_S$, the restriction of A to S, which equals as a sequence

$$A(S) = (i_A)_{i \in S}.$$

Since the bifurcation of a triangle does not depend on the ordering of the points, the bifurcations of a quadrilateral can be written as

$$\mathfrak{b}(i)_A = \mathfrak{b}\big(A(\{1, 2, 3, 4\} - i)\big)$$

for all $i \in \{1, 2, 3, 4\}$.

5.1 Definition (Isometric Image of a Quadrilateral)**.** For every quadrilateral A and each isometry $h \in \mathrm{Is}\,(\mathcal{T})$ we define

$$h(A) := \big(h(1_A), h(2_A), h(3_A), h(4_A)\big),$$

that is $i_{h(A)} = h(i_A)$ for all $i \in \{1,2,3,4\}$. Clearly $h(A)$ is a quadrilateral by injectivity of isometries. This definition can be rewritten in the interpretation of A as a map like $h(A) = h \circ A$, which allows for an action of $\mathrm{Is}\,(\mathcal{T})$ on the space of quadrilaterals, if $\mathrm{Is}\,(\mathcal{T})$ is a group.

5.2 Proposition. *For each quadrilateral $A \in \mathcal{T}_{\mathrm{Q}}$ and each isometry $h \in \mathrm{Is}\,(\mathcal{T})$ holds $\mathfrak{b}(i)_{h(A)} = h\big(\mathfrak{b}(i)_A\big)$ for $i \in \{1,2,3,4\}$.*

Proof. One has

$$\begin{aligned} \mathfrak{b}(i)_{h(A)} &= \mathfrak{b}\big(h(A)(\{1,2,3,4\} - i)\big) = \mathfrak{b}\big(h \circ A(\{1,2,3,4\} - i)\big) \\ &\overset{\text{Prop. 4.8}}{=} h\Big(\mathfrak{b}\big(A(\{1,2,3,4\} - i)\big)\Big) \\ &= h\big(\mathfrak{b}(i)_A\big). \end{aligned}$$

□

5.3 Corollary. *For each isometry $h \in \mathrm{Is}\,(\mathcal{T})$ and each quadrilateral $A \in \mathcal{T}_{\mathrm{Q}}$ holds $\mathrm{d}\Big(\mathfrak{b}(i)_{h(A)}, \mathfrak{b}(j)_{h(A)}\Big) = \mathrm{d}\Big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\Big)$ for $i, j \in \{1,2,3,4\}$.*

Proof. One has

$$\begin{aligned} \mathrm{d}\Big(\mathfrak{b}(i)_{h(A)}, \mathfrak{b}(j)_{h(A)}\Big) &\overset{\text{Prop. 5.2}}{=} \mathrm{d}\Big(h\big(\mathfrak{b}(i)_A\big), h\big(\mathfrak{b}(j)_A\big)\Big) \\ &= \mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big). \end{aligned}$$

□

5.4 Definition (Action of S_4 on the Space of Quadrilaterals)**.** For any permutation $\sigma \in S_4$ and any quadrilateral $A \in \mathcal{T}_{\mathrm{Q}}$, a quadrilateral

$$\sigma(A) := \big(\sigma^{-1}(1)_A, \sigma^{-1}(2)_A, \sigma^{-1}(3)_A, \sigma^{-1}(4)_A\big) \in \mathcal{T}_{\mathrm{Q}}$$

is defined. The definition can be written as $i_{\sigma(A)} = \sigma^{-1}(i)_A$ for all $i \in \{1,2,3,4\}$. In terms of A as a map, the assignment reads $\sigma(A) = A \circ \sigma^{-1}$ and defines therefore an action of S_4 on the space of quadrilaterals.

5.5 Proposition. *For all quadrilaterals $A \in \mathcal{T}_{\mathrm{Q}}$ and all permutations $\sigma \in S_4$ holds $\mathfrak{b}(i)_{\sigma(A)} = \mathfrak{b}\big(\sigma^{-1}(i)\big)_A$ for all $i \in \{1,2,3,4\}$.*

Proof. One has

$$\begin{aligned}\mathfrak{b}\big(\sigma^{-1}(i)\big)_A &= \mathfrak{b}\Big(A\big(\{1,2,3,4\}-\sigma^{-1}(i)\big)\Big)\\ &= \mathfrak{b}\Big(A\big(\sigma^{-1}\{1,2,3,4\}-\sigma^{-1}(i)\big)\Big) = \mathfrak{b}\big(A\circ\sigma^{-1}(\{1,2,3,4\}-i)\big)\\ &= \mathfrak{b}(i)_{\sigma(A)}.\end{aligned}$$

□

5.6 Proposition. *Isometries $h \in \mathrm{Is}(\mathcal{T})$ commute with the action of permutations $\sigma \in S_4$ on quadrilaterals $A \in \mathcal{T}_{\mathrm{Q}}$ in the sense that $\sigma \circ h(A) = h \circ \sigma(A)$.*

Proof. For all $i \in \{1,2,3,4\}$ holds

$$\begin{aligned}i_{\sigma\circ h(A)} &= \sigma^{-1}(i)_{h(A)} = h\big(\sigma^{-1}(i)_A\big)\\ &= h\big(i_{\sigma(A)}\big) = i_{h\circ\sigma(A)}.\end{aligned}$$

□

5.2 The Klein Type of a Quadrilateral

In this section we provide a partition of the set of quadrilaterals $\mathcal{T}_{\mathrm{Q}}$ into four classes depending on equalities of the four bifurcations. For an easier notation the indices $1,2,3,4$ of a quadrilateral A will be confused with the objects $1_A, 2_A, 3_A, 4_A \in \mathcal{T} \cup \mathcal{T}(\infty)$. The section is concluded with an investigation of how a permutation of the points of a quadrilateral and a mapping by an isometry affects the classes of quadrilaterals.

5.7 Lemma. *If $(1,2,3,4)$ is a quadrilateral and $\mathfrak{b}(2,3,4) = \mathfrak{b}(1,3,4)$ then $\mathfrak{b}(1,2,4) = \mathfrak{b}(1,2,3)$.*

Proof. If we take $z = \mathfrak{b}(2,3,4) = \mathfrak{b}(1,3,4)$ then $[z,2\rangle \subset \langle 3,2\rangle$ and $[z,1\rangle \subset \langle 3,1\rangle$. This shows $(z,2,1) < (3,2,1)$. On the other hand one has $[z,2\rangle \subset \langle 4,2\rangle$ and $[z,1\rangle \subset \langle 4,1\rangle$ and therefore $(z,2,1) < (4,2,1)$. Altogether, from Proposition 4.6 follows $\mathfrak{b}(3,2,1) = \mathfrak{b}(z,2,1) = \mathfrak{b}(4,2,1)$. □

5.8 Proposition. *We assume that $A = (1,2,3,4)$ is a quadrilateral. In the case that $\mathfrak{b}(2,3,4) \neq \mathfrak{b}(1,3,4)$, all four bifurcations of A lie on the line $\langle 2,1\rangle$. Moreover $\langle 2,1\rangle = \big\langle 2,\mathfrak{b}(2,3,4)\big]\big[\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)\big]\big[\mathfrak{b}(1,3,4),1\big\rangle$.*

Proof. Both vertices $\mathfrak{b}(2,3,4)$ and $\mathfrak{b}(1,3,4)$ lie on $\langle 3,4\rangle$. After possibly relabeling 3 and 4 we can assume that

$$(*) \qquad \langle 3,\mathfrak{b}(2,3,4)][\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)][\mathfrak{b}(1,3,4),4\rangle$$

is a line. By $(*)$, $[\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)][\mathfrak{b}(1,3,4),4\rangle$ is a line. The composition $\langle 2,\mathfrak{b}(2,3,4)][\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)][\mathfrak{b}(1,3,4),4\rangle$ is a line by the definition of the bifurcation $\mathfrak{b}(2,3,4)$. This shows that $\langle 2,\mathfrak{b}(2,3,4)][\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)]$ is a line.

On the other hand, $(*)$ implies that $\langle 3,\mathfrak{b}(2,3,4)][\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)]$ is a line. The composition $\langle 3,\mathfrak{b}(2,3,4)][\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)][\mathfrak{b}(1,3,4),1\rangle$ is a line by the definition of the bifurcation $\mathfrak{b}(1,3,4)$. Thus $[\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)][\mathfrak{b}(1,3,4),1\rangle$ is a line.

Since the length of the segment $\big[\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)\big]$ is positive, the whole composition

$$\langle 2,\mathfrak{b}(2,3,4)][\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)][\mathfrak{b}(1,3,4),1\rangle$$

is a line, hence equals $\langle 2,1\rangle$. This shows $\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)\in\langle 2,1\rangle$. The remaining bifurcations lie on $\langle 2,1\rangle$ by definition. □

Theorem 2. *Every quadrilateral has two pairs of coinciding bifurcations.*

Proof. In the case that three bifurcations of a quadrilateral $A=(1,2,3,4)$ are mutually distinct, we may assume after possibly relabeling symbols, that

$$\begin{aligned} \mathfrak{b}(2,3,4) &\neq \mathfrak{b}(1,3,4),\\ \mathfrak{b}(1,3,4) &\neq \mathfrak{b}(1,2,4),\\ \text{and}\quad \mathfrak{b}(1,2,4) &\neq \mathfrak{b}(2,3,4). \end{aligned}$$

An application of Proposition 5.8 to the three inequalities shows that all four bifurcations lie on each of the lines

$$\langle 2,1\rangle,\quad \langle 3,2\rangle \quad\text{and}\quad \langle 1,3\rangle.$$

Now the Bifurcation Lemma states that these four bifurcations are all equal to $\mathfrak{b}(1,2,3)$. This is a contradiction.

It is clear now that the bifurcations of A form a set of vertices with cardinality at most two. It follows from the "pigeon hole principle" [Aig97] that at least two of them must be equal. Lemma 5.7 concludes that the remaining two are also equal. □

5.9 Definition (Centered Quadrilaterals)**.** A quadrilateral is called *centered*, if all its bifurcations are equal.

5.10 Corollary. *For each quadrilateral $A \in \mathcal{T}_{\mathrm{Q}}$ holds exactly one of the following conditions.*

(a) $\mathfrak{b}(4)_A = \mathfrak{b}(3)_A \neq \mathfrak{b}(2)_A = \mathfrak{b}(1)_A$

(b) $\mathfrak{b}(4)_A = \mathfrak{b}(2)_A \neq \mathfrak{b}(3)_A = \mathfrak{b}(1)_A$

(c) $\mathfrak{b}(4)_A = \mathfrak{b}(1)_A \neq \mathfrak{b}(3)_A = \mathfrak{b}(2)_A$

(d) *A is centered.*

Proof. If not all four bifurcations coincide (case (d)), then by Theorem 2 exactly one of the remaining bifurcations equals $\mathfrak{b}(4)_A$ (cases (a) to (c)). The two other bifurcations are equal among themselves. □

5.11 Definition (Bifurcation Type)**.** A quadrilateral satisfying case (a) of Corollary 5.10 is called a quadrilateral of *type* (a). The analogue definitions are set for the remaining cases of the Corollary. The condition that two quadrilaterals have the same type is clearly an equivalence relation on any set $S \subset \mathcal{T}_{\mathrm{Q}}$ of quadrilaterals. The equivalence classes are called *type classes*. The collection of type classes of S is denoted by $\mathrm{T}_{\mathfrak{b}}(S)$. We write

$$\pi_{\mathfrak{b}} : S \longrightarrow \mathrm{T}_{\mathfrak{b}}(S)$$

for the projection to the type class.

Once written the classification for quadrilaterals in Corollary 5.10 it will turn out that the action of S_4 on quadrilaterals is precisely described by the Klein 4-group. The Klein 4-group V is the subgroup of S_4 that consists of the elements $V = \{\mathrm{Id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ in cycle notation. See Appendix A for further information.

5.12 Proposition. *For sets of quadrilaterals $S \subset \mathcal{T}_{\mathrm{Q}}$ there is a map*

$$\mathrm{Kl} : S \longrightarrow V$$

defined as $\mathrm{Kl}(A) := \mathrm{Id}$ *if $A \in S$ is centered. If A is not centered then* $\mathrm{Kl}(A) := (j\,k)(l\,m)$ *for pairwise distinct numbers $j, k, l, m \in \{1, 2, 3, 4\}$ such that $\mathfrak{b}(j)_A = \mathfrak{b}(k)_A$ and $\mathfrak{b}(l)_A = \mathfrak{b}(m)_A$.*

bifurcation type	`(a)`	`(b)`	`(c)`	`(d)`
Klein type	$(1\,2)(3\,4)$	$(1\,3)(2\,4)$	$(1\,4)(2\,3)$	$\mathrm{Id}\|_V$

Figure 5.1: The correspondence between the bifurcation type and the Klein type of quadrilaterals.

The function Kl *induces an injective function* $\widehat{\mathrm{Kl}} : \mathrm{T}_{\mathfrak{b}}(S) \to V$ *such that* $\widehat{\mathrm{Kl}} \circ \pi_{\mathfrak{b}}(A) = \mathrm{Kl}(A)$.

Proof. We show first that Kl is well defined. For centered quadrilaterals this is clear. If $A \in S$ is of type `(a)` then it satisfies by definition the relations

$$\mathfrak{b}(4)_A = \mathfrak{b}(3)_A \neq \mathfrak{b}(2)_A = \mathfrak{b}(1)_A.$$

The possible values for $\mathrm{Kl}(A)$ would be $(1\,2)(3\,4)$, $(2\,1)(3\,4)$, $(1\,2)(4\,3)$ and $(2\,1)(4\,3)$. They all coincide as permutations, so there is a unique group element $\mathrm{Kl}(A) \in V$ assigned to A. Similarly $\mathrm{Kl}(A) = (1\,3)(2\,4)$ for all quadrilaterals A of type `(b)` and $\mathrm{Kl}(A) = (1\,4)(2\,3)$ for all quadrilaterals A of type `(c)`.

Since the conditions on a quadrilateral A for the assignment of $\mathrm{Kl}(A)$ only depend on the bifurcation type of A, Kl is constant on the equivalence classes $\mathrm{T}_{\mathfrak{b}}(S)$. Thus a function $\widehat{\mathrm{Kl}}\big(\pi_{\mathfrak{b}}(A)\big) := \mathrm{Kl}(A)$ can be defined. Obviously the value $\mathrm{Kl}(A)$ determines the bifurcation type of A. Thus $\mathrm{Kl}(A) = \mathrm{Kl}(B)$ implies $\pi_{\mathfrak{b}}(A) = \pi_{\mathfrak{b}}(B)$. So $\pi_{\mathfrak{b}}(A) \mapsto \mathrm{Kl}(A) = \widehat{\mathrm{Kl}}\big(\pi_{\mathfrak{b}}(A)\big)$ is injective. □

5.13 Definition (Klein Type)**.** The permutation $\mathrm{Kl}(A) \in V$ assigned in Proposition 5.12 is called *Klein type* of a quadrilateral $A \in \mathcal{T}_{\mathrm{Q}}$. Note that $\widehat{\mathrm{Kl}}$ is invertible on its image so there is a correspondence between the bifurcation type and the Klein type of quadrilaterals as displayed in Figure 5.1.

Images of Quadrilaterals under Isometries

The Klein type is invariant under isometries.

5.14 Corollary. *For all quadrilaterals* $A \in \mathcal{T}_{\mathrm{Q}}$ *and all isometries* $h \in \mathrm{Is}\,(\mathcal{T})$ *holds* $\mathrm{Kl}\big(h(A)\big) = \mathrm{Kl}(A)$.

Proof. By Corollary 5.3 holds $\mathrm{d}\big(\mathfrak{b}(i)_{h(A)}, \mathfrak{b}(j)_{h(A)}\big) = \mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big)$ for all $i, j \in \{1, 2, 3, 4\}$. Hence the bifurcation types of A and $h(A)$ are the same. So are the Klein types by definition. $\square$

Action of S_4 on Quadrilaterals

Unlike under a mapping by an isometry, the type of a quadrilateral is changed by the action of S_4 on the points of the quadrilateral.

5.15 Definition. A set S of quadrilaterals is called *closed under* S_4 if $S_4(S) = \{\sigma(s) : \sigma \in S_4, s \in S\} = S$.

5.16 Note. The focus on spaces of quadrilaterals S that are closed under S_4 is not a restriction on the applicability of the following results. The assumption must by taken if a map $s \mapsto \sigma(s)$ shall be defined for $s \in S$ and $\sigma \in S_4$. Note also that $\mathcal{T}_{\mathrm{Q}}$ is closed under S_4 since the group acts on $\mathcal{T}_{\mathrm{Q}}$. The reason for an emphasize on subsets $S \subset \mathcal{T}_{\mathrm{Q}}$ is the aim of working with quadrilaterals of boundary points in Chapter 9. These spaces are closed under S_4, too.

Recall from Proposition A.2 that S_4 acts on V by conjugation.

$$(\sigma, \tau) \longmapsto \alpha\big(\sigma\big)(\tau) := \sigma\tau\sigma^{-1} \tag{5.1}$$

for all $\sigma \in S_4, \tau \in V$.

Theorem 3. *If* $S \subset \mathcal{T}_{\mathrm{Q}}$ *is closed under* S_4 *then for all* $\sigma \in S_4$ *the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\mathrm{Kl}} & V \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\alpha(\sigma)} \\ S & \xrightarrow[\mathrm{Kl}]{} & V \end{array}$$

commutes, that is $\mathrm{Kl} \circ \sigma = \alpha(\sigma) \circ \mathrm{Kl}$.

Proof. Assume that $A \in S$ is a quadrilateral and choose $\sigma \in S_4$. In the case that A is centered then $\mathrm{Kl}(A) = \mathrm{Id}$ and $\sigma(A)$ is centered, i.e. $\mathrm{Kl}(\sigma A) = \mathrm{Kl}(A) = \mathrm{Id}$. Thus $\alpha(\sigma) \circ \mathrm{Kl}(A) = \alpha(\sigma)(\mathrm{Id}) = \sigma \mathrm{Id} \sigma^{-1} = \mathrm{Id} = \mathrm{Kl} \circ \sigma(A)$.

In the case that A is not centered, then for some $\tau \in S_4$ there are relations among the bifurcations of A

$$\mathfrak{b}\big(\tau(4)\big)_A = \mathfrak{b}\big(\tau(3)\big)_A \neq \mathfrak{b}\big(\tau(2)\big)_A = \mathfrak{b}\big(\tau(1)\big)_A.$$

(Confer Corollary 5.10.) This shows that $\mathrm{Kl}(A) = \big(\tau(4)\,\tau(3)\big)\big(\tau(2)\,\tau(1)\big)$. An application of Proposition 5.5 gives

$$\mathfrak{b}\big(\sigma\tau(4)\big)_{\sigma(A)} = \mathfrak{b}\big(\sigma\tau(3)\big)_{\sigma(A)} \neq \mathfrak{b}\big(\sigma\tau(2)\big)_{\sigma(A)} = \mathfrak{b}\big(\sigma\tau(1)\big)_{\sigma(A)}.$$

A comparison gives

$$\begin{array}{rcl} \mathrm{Kl}\circ\sigma(A) & = & \big(\sigma\tau(4)\,\sigma\tau(3)\big)\big(\sigma\tau(2)\,\sigma\tau(1)\big) \\ & \overset{\text{Prop. A.2}}{=} & \sigma\big(\tau(4)\,\tau(3)\big)\big(\tau(2)\,\tau(1)\big)\sigma^{-1} \quad = \quad \sigma\mathrm{Kl}(A)\sigma^{-1} \\ & = & \alpha(\sigma)\circ\mathrm{Kl}(A). \end{array}$$

□

5.17 Corollary. *The Klein type is invariant under the Klein 4-group, i.e.* $\mathrm{Kl}(\sigma A) = \mathrm{Kl}(A)$ *for all quadrilaterals* $A \in \mathcal{T}_{\mathrm{Q}}$ *and all* $\sigma \in V$.

Proof. We apply Theorem 3 to the set $S = \mathcal{T}_{\mathrm{Q}}$. For $\sigma \in V$ and $A \in \mathcal{T}_{\mathrm{Q}}$ holds $\mathrm{Kl}\big(\sigma(A)\big) = \alpha(\sigma)\big(\mathrm{Kl}(A)\big) = \sigma\mathrm{Kl}(A)\sigma^{-1} = \sigma\sigma^{-1}\mathrm{Kl}(A) = \mathrm{Kl}(A)$ since V is abelian. □

5.18 Lemma. *If* $A \in \mathcal{T}_{\mathrm{Q}}$, *then for* $\sigma = (2\,3\,4)$ *holds*

$$\mathrm{Kl}(\sigma A) = \begin{cases} (1\,3)(2\,4) & \text{if } \mathrm{Kl}(A) = (1\,2)(3\,4) \\ (1\,4)(2\,3) & \text{if } \mathrm{Kl}(A) = (1\,3)(2\,4) \\ (1\,2)(3\,4) & \text{if } \mathrm{Kl}(A) = (1\,4)(2\,3). \end{cases}$$

In particular, if $S \subset \mathcal{T}_{\mathrm{Q}}$ *is closed under* S_4 *and* S *has a non-centered quadrilateral then it has quadrilaterals of all three non-centered types.*

Proof. This follows directly from Theorem 3. □

5.19 Proposition. *If* $S \subset \mathcal{T}_{\mathrm{Q}}$ *is closed under* S_4 *then* S_4 *acts on* $\mathrm{T}_{\mathfrak{b}}(S)$ *by* $(\sigma, C) \longmapsto \beta(\sigma)(C)$ *for* $\sigma \in S_4$ *and* $C \in \mathrm{T}_{\mathfrak{b}}(S)$ *such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\pi_{\mathfrak{b}}} & \mathrm{T}_{\mathfrak{b}}(S) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\beta(\sigma)} \\ S & \xrightarrow[\pi_{\mathfrak{b}}]{} & \mathrm{T}_{\mathfrak{b}}(S) \end{array}$$

commutes, that is $\beta(\sigma)\circ\pi_{\mathfrak{b}} = \pi_{\mathfrak{b}}\circ\sigma$.

Proof. For $\sigma \in S_4$ a map $\beta(\sigma) : \mathrm{T}_{\mathfrak{b}}(S) \to \mathrm{T}_{\mathfrak{b}}(S)$ is defined by $\beta(\sigma)\big(\pi_{\mathfrak{b}}(A)\big) := \pi_{\mathfrak{b}}\big(\sigma(A)\big)$. To prove the validity of this definition, it has to be verified for any two quadrilaterals $A, B \in S$ that $\pi_{\mathfrak{b}}(A) = \pi_{\mathfrak{b}}(B)$ implies $\pi_{\mathfrak{b}}(\sigma A) = \pi_{\mathfrak{b}}(\sigma B)$. Let $\sigma \in S_4$ and $A, B \in S$. The equality $\pi_{\mathfrak{b}}(A) = \pi_{\mathfrak{b}}(B)$ implies $\widehat{\mathrm{Kl}} \circ \pi_{\mathfrak{b}}(A) = \widehat{\mathrm{Kl}} \circ \pi_{\mathfrak{b}}(B)$, which is equivalent to $\mathrm{Kl}(A) = \mathrm{Kl}(B)$. Theorem 3 shows that $\mathrm{Kl}(\sigma A) = \sigma \mathrm{Kl}(A)\sigma^{-1} = \sigma \mathrm{Kl}(B)\sigma^{-1} = \mathrm{Kl}(\sigma B)$, which is equivalent to $\widehat{\mathrm{Kl}} \circ \pi_{\mathfrak{b}}(\sigma A) = \widehat{\mathrm{Kl}} \circ \pi_{\mathfrak{b}}(\sigma B)$. By Proposition 5.12 the function $\widehat{\mathrm{Kl}}$ is injective, so that $\pi_{\mathfrak{b}}(\sigma A) = \pi_{\mathfrak{b}}(\sigma B)$.

It remains to show that β is a homomorphism of groups. This implies then bijectivity of $\beta(\sigma)$ for all $\sigma \in S_4$, so that β is indeed a homomorphism to the group $\mathrm{Perm}\big(\mathrm{T}_{\mathfrak{b}}(S)\big)$. We choose two group elements $\sigma, \tau \in S_4$. Then for any $A \in S$ holds $\beta(\sigma)\beta(\tau)\big(\pi_{\mathfrak{b}}(A)\big) = \beta(\sigma)\pi_{\mathfrak{b}}\big(\tau(A)\big) = \pi_{\mathfrak{b}}\big(\sigma\tau(A)\big) = \beta(\sigma\tau)\big(\pi_{\mathfrak{b}}(A)\big)$ because S_4 acts on S. □

5.20 Corollary. *If a set $S \subset \mathcal{T}_{\mathrm{Q}}$ is closed under S_4 then for all $\sigma \in S_4$ the diagram*

$$\begin{array}{ccc} \mathrm{T}_{\mathfrak{b}}(S) & \xrightarrow{\widehat{\mathrm{Kl}}} & V \\ {\scriptstyle \beta(\sigma)}\big\downarrow & & \big\downarrow{\scriptstyle \alpha(\sigma)} \\ \mathrm{T}_{\mathfrak{b}}(S) & \xrightarrow[\widehat{\mathrm{Kl}}]{} & V \end{array}$$

commutes, i.e. $\alpha(\sigma) \circ \widehat{\mathrm{Kl}} = \widehat{\mathrm{Kl}} \circ \beta(\sigma)$.

Proof. For all quadrilaterals $A \in S$ holds

$$\begin{aligned} \alpha(\sigma) \circ \widehat{\mathrm{Kl}}\big(\pi_{\mathfrak{b}}(A)\big) &\overset{\text{Prop. 5.12}}{=} \alpha(\sigma) \circ \mathrm{Kl}(A) \overset{\text{Theorem 3}}{=} \mathrm{Kl} \circ \sigma(A) \\ &\overset{\text{Prop. 5.12}}{=} \widehat{\mathrm{Kl}} \circ \pi_{\mathfrak{b}} \circ \sigma(A) \overset{\text{Prop. 5.19}}{=} \widehat{\mathrm{Kl}} \circ \beta(\sigma)\big(\pi_{\mathfrak{b}}(A)\big). \end{aligned}$$

□

5.21 Corollary. *If $S \subset \mathcal{T}_{\mathrm{Q}}$ is closed under S_4 then* $\ker(\beta) = V$ *if and only if S has a non-centered quadrilateral. Otherwise* $\ker(\beta) = S_4$.

Proof. For $\sigma \in S_4$, the condition $\sigma \in \ker(\beta)$ is equivalent to $\beta(\sigma)\big(\pi_{\mathfrak{b}}(A)\big) = \pi_{\mathfrak{b}}(A)$ for all $A \in S$. By injectivity of $\widehat{\mathrm{Kl}}$ (Prop. 5.12) this is equivalent to $\widehat{\mathrm{Kl}} \circ \beta(\sigma) \circ \pi_{\mathfrak{b}}(A) = \widehat{\mathrm{Kl}} \circ \pi_{\mathfrak{b}}(A)$ for all $A \in S$. By Theorem 3 this is equivalent to the condition

$$\alpha(\sigma)\big(\mathrm{Kl}(A)\big) \quad = \quad \mathrm{Kl}(A)$$

for all $A \in S$. Two cases shall be distinguished. If S has a non-centered quadrilateral then it has quadrilaterals of all three non-centered types (Lemma 5.18). So the last condition is equivalent to the condition $\alpha(\sigma)(a) = a$ for all $a \in V$ (note that $\alpha(\sigma)(\mathrm{Id}_V) = \sigma \mathrm{Id}_V \sigma^{-1} = \sigma\sigma^{-1} = \mathrm{Id}_V$ holds for all $\sigma \in S_4$.) So this is equivalent to $\alpha(\sigma) = \mathrm{Id}_{\mathrm{Aut}(V)}$ which is true if and only if $\sigma \in \ker(\alpha) = V$ (Prop. A.3).

In the case that all quadrilaterals $A \in S$ are centered then above condition is equivalent to $\alpha(\sigma)(\mathrm{Id}_V) = \mathrm{Id}_V$, which is true for all $\sigma \in S_4$. □

5.3 The Inner Diameter of a Quadrilateral

The *inner diameter* of a quadrilateral A is defined as the integer number

$$\mathtt{diam}(A) := \max_{i,j \in \{1,2,3,4\}} \mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big).$$

5.22 Proposition. *If A is a quadrilateral then*

- $\Big(\mathtt{diam}(A) = 0\Big) \iff \Big(A \textit{ is centered}\Big)$.
- *If* $\mathtt{diam}(A) > 0$ *then for all* $i, j \in \{1, 2, 3, 4\}$ *holds* $\Big(\mathfrak{b}(i)_A \neq \mathfrak{b}(j)_A\Big) \iff \Big(\mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big) = \mathtt{diam}(A)\Big)$.

Proof. The first assertion is trivial. We assume that $\mathtt{diam}(A) > 0$. In the case $\mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big) = \mathtt{diam}(A) > 0$ one has $\mathfrak{b}(i)_A \neq \mathfrak{b}(j)_A$. Conversely, if $\mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big) < \mathtt{diam}(A)$ then there is a pair k, l of numbers with $\mathfrak{b}(k)_A \neq \mathfrak{b}(l)_A$ such that

$$\mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big) < \mathrm{d}\big(\mathfrak{b}(k)_A, \mathfrak{b}(l)_A\big).$$

In the case that $\mathfrak{b}(i)_A \neq \mathfrak{b}(j)_A$ we can assume by Theorem 2 that $\mathfrak{b}(i)_A = \mathfrak{b}(k)_A$ (after possibly relabeling k and l). This implies that $\mathfrak{b}(j)_A = \mathfrak{b}(l)_A$ and contradicts above inequality. □

5.23 Corollary. *For each isometry $h \in \mathrm{Is}\,(\mathcal{T})$ and each quadrilateral A holds* $\mathtt{diam}\big(h(A)\big) = \mathtt{diam}(A)$.

Proof. This follows directly from the definition of the inner diameter and Corollary 5.3. □

5.24 Corollary. *For all $\sigma \in S_4$ and all $A \in \mathcal{T}_{\mathrm{Q}}$ holds* $\mathtt{diam}\big(\sigma(A)\big) = \mathtt{diam}(A)$.

Proof. If A is centered then $\sigma(A)$ is centered and both diameters are zero. If A is not centered then $\mathfrak{b}(i)_A \neq \mathfrak{b}(j)_A$ for two $i,j \in \{1,2,3,4\}$. Since by Proposition 5.5 holds $\mathfrak{b}\big(\sigma(k)\big)_{\sigma(A)} = \mathfrak{b}(i)_A$ for all k, Proposition 5.22 gives $\mathtt{diam}\big(\sigma(A)\big) = \mathrm{d}\big(\mathfrak{b}\big(\sigma(i)\big)_{\sigma(A)}, \mathfrak{b}\big(\sigma(j)\big)_{\sigma(A)}\big) = \mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big) = \mathtt{diam}(A)$. □

5.25 Corollary. *For all quadrilaterals $A \in \mathcal{T}_{\mathrm{Q}}$ and all $\sigma \in S_4$ holds the equation* $\mathrm{d}\Big(\mathfrak{b}\big(\sigma(1)\big)_A, \mathfrak{b}\big(\sigma(2)\big)_A\Big) = \mathrm{d}\Big(\mathfrak{b}\big(\sigma(3)\big)_A, \mathfrak{b}\big(\sigma(4)\big)_A\Big)$.

Proof. Since $\mathfrak{b}\big(\sigma(i)\big)_A = \mathfrak{b}(i)_{\sigma^{-1}(A)}$ for $i = 1,2,3,4$ by Prop. 5.5, it remains to show that $\mathrm{d}\big(\mathfrak{b}(1)_B, \mathfrak{b}(2)_B\big) = \mathrm{d}\big(\mathfrak{b}(3)_B, \mathfrak{b}(4)_B\big)$ for the quadrilateral $B := \sigma^{-1}(A)$. From Theorem 2 follows that $\mathfrak{b}(1)_B = \mathfrak{b}(2)_B \Rightarrow \mathfrak{b}(3)_B = \mathfrak{b}(4)_B$ and that $\mathfrak{b}(1)_B \neq \mathfrak{b}(2)_B \Rightarrow \mathfrak{b}(3)_B \neq \mathfrak{b}(4)_B$. In the second case Proposition 5.22 gives $\mathrm{d}\big(\mathfrak{b}(1)_B, \mathfrak{b}(2)_B\big) = \mathtt{diam}(B) = \mathrm{d}\big(\mathfrak{b}(3)_B, \mathfrak{b}(4)_B\big)$. □

5.4 Alternate Aspects of the Klein Type

A second and a third description of the Klein type of quadrilaterals are set up. They will smooth the way for a description of the Klein type by integers.

Theorem 4. *If $(1,2,3,4) \in \mathcal{T}_{\mathrm{Q}}$ is a quadrilateral and $\langle 1,2\rangle \cap \langle 3,4\rangle \neq \emptyset$ then $\langle 1,2\rangle \cap \langle 3,4\rangle = \big[\mathfrak{b}(1,3,4), \mathfrak{b}(2,3,4)\big]$ as an equality of vertex sets.*

Proof. We show that $\langle 1,2\rangle \cap \langle 3,4\rangle \subset \big[\mathfrak{b}(1,3,4), \mathfrak{b}(2,3,4)\big]$. Let $c \in \langle 1,2\rangle \cap \langle 3,4\rangle$ and assume that $c \notin \big[\mathfrak{b}(1,3,4), \mathfrak{b}(2,3,4)\big]$. Then by Corollary 3.3 one of the cases $c \in \big[\mathfrak{b}(1,3,4), 3\big\rangle \cap \big[\mathfrak{b}(2,3,4), 3\big\rangle$ or $c \in \big[\mathfrak{b}(1,3,4), 4\big\rangle \cap \big[\mathfrak{b}(2,3,4), 4\big\rangle$ is true. Since the statement does not change under a relabeling $3 \leftrightarrow 4$ we can assume that $c \in \big[\mathfrak{b}(1,3,4), 3\big\rangle$ and $c \in \big[\mathfrak{b}(2,3,4), 3\big\rangle$. From the first inclusion follows $c \in \langle 1,3\rangle$, from the second one $c \in \langle 2,3\rangle$. Since we took $c \in \langle 1,2\rangle$ it follows that $c = \mathfrak{b}(1,2,3)$. The assumption $c \notin \big[\mathfrak{b}(1,3,4), \mathfrak{b}(2,3,4)\big]$ implies $c \neq \mathfrak{b}(1,3,4)$ and $c \neq \mathfrak{b}(2,3,4)$ and by Theorem 2 follows $c = \mathfrak{b}(1,2,4)$. This shows that $c \in \langle 1,4\rangle$. Together with $c \in \langle 1,3\rangle$ and $c \in \langle 3,4\rangle$ follows the contradiction $c = \mathfrak{b}(1,3,4)$.

We show that $\big[\mathfrak{b}(1,3,4), \mathfrak{b}(2,3,4)\big] \subset \langle 1,2\rangle \cap \langle 3,4\rangle$ if $\langle 1,2\rangle \cap \langle 3,4\rangle \neq \emptyset$. To this end we prove first that $\big(\langle 1,2\rangle \cap \langle 3,4\rangle \neq \emptyset\big) \Rightarrow \big(\mathfrak{b}(1,3,4) \in \langle 1,2\rangle\big)$. Then $\langle 1,2\rangle = \langle 2,1\rangle$ as a vertex set implies that both bifurcations $\mathfrak{b}(1,3,4), \mathfrak{b}(2,3,4)$

lie on $\langle 1,2\rangle$. Thus $\big[\mathfrak{b}(1,3,4),\mathfrak{b}(2,3,4)\big] \subset \langle 1,2\rangle$ by uniqueness of geodesics. The inclusion $\big[\mathfrak{b}(1,3,4),\mathfrak{b}(2,3,4)\big] \subset \langle 3,4\rangle$ is obvious.

In the case that $\mathfrak{b}(1,3,4) \notin \langle 1,2\rangle$ Theorem 2 gives $\mathfrak{b}(1,3,4) = \mathfrak{b}(2,3,4)$ and $\mathfrak{b}(1,2,3) = \mathfrak{b}(1,2,4)$, because the last mentioned bifurcations lie on $\langle 1,2\rangle$. From the first part of this theorem follows $\langle 1,2\rangle \cap \langle 3,4\rangle \subset S := \big[\mathfrak{b}(1,3,4),\mathfrak{b}(2,3,4)\big]$ and $\langle 3,4\rangle \cap \langle 1,2\rangle \subset T := \big[\mathfrak{b}(1,2,3),\mathfrak{b}(1,2,4)\big]$ as an equation of vertex sets. Since both S and T have cardinality one they are equal. This shows that $(1,2,3,4)$ is centered. In particular $\mathfrak{b}(1,3,4) = \mathfrak{b}(1,2,3) \in \langle 1,2\rangle$. □

5.26 Corollary. *We assume that* $(1,2,3,4) \in \mathcal{T}_{\mathrm{Q}}$. *Then*

1) $\langle 1,2\rangle \cap \langle 3,4\rangle = \emptyset \quad \Leftrightarrow \quad \mathfrak{b}(1,2,3) = \mathfrak{b}(1,2,4) \neq \mathfrak{b}(1,3,4) = \mathfrak{b}(2,3,4).$

2) $\langle 1,2\rangle \cap \langle 3,4\rangle \neq \emptyset$

$$\Leftrightarrow \quad \langle 1,2\rangle \cap \langle 3,4\rangle = \big[\mathfrak{b}(1,3,4),\mathfrak{b}(2,3,4)\big] = \big[\mathfrak{b}(1,2,3),\mathfrak{b}(1,2,4)\big]$$

as a vertex set

$$\Leftrightarrow \quad \big[\mathfrak{b}(1,3,4),\mathfrak{b}(2,3,4)\big] = \big[\mathfrak{b}(1,2,3),\mathfrak{b}(1,2,4)\big]$$

as a vertex set

$$\Leftrightarrow \quad \big\{\mathfrak{b}(1,3,4),\mathfrak{b}(2,3,4)\big\} = \big\{\mathfrak{b}(1,2,3),\mathfrak{b}(1,2,4)\big\}.$$

3) $\big|\langle 1,2\rangle \cap \langle 3,4\rangle\big| = 1 \quad \Leftrightarrow \quad (1,2,3,4)$ *is centered.*

4) $\big|\langle 1,2\rangle \cap \langle 3,4\rangle\big| \geq 2$

$$\Leftrightarrow \quad \mathfrak{b}(1,3,4) \neq \mathfrak{b}(2,3,4) \quad \Leftrightarrow \quad \mathfrak{b}(1,2,3) \neq \mathfrak{b}(1,2,4).$$

Proof. Theorem 2 shows "⇒" for 1) because $\mathfrak{b}(1,2,3),\mathfrak{b}(1,2,4) \in \langle 1,2\rangle$ and $\mathfrak{b}(1,3,4),\mathfrak{b}(2,3,4) \in \langle 3,4\rangle$. The first implication "⇒" for 2) comes from Theorem 4. The following two implications "⇒" for 2) are obvious. The first statement F_1 of 1) and the first statement F_2 of 2) are alternatives. The second statement L_1 of 1) and the last statement L_2 of 2) are contradictorily. Thus one can prove $L_1 \Rightarrow F_1$ by contradiction

$$(L_1 \wedge \neg F_1) \Rightarrow (L_1 \wedge F_2) \Rightarrow (L_1 \wedge L_2).$$

Similarly $L_2 \Rightarrow F_2$ is proved

The implication "⇒" for 3) follows from part 2) because the segments between the bifurcations are equal and have length zero. If $(1,2,3,4)$ is centered

Kl(A)	bifurcation condition	intersection condition
$(1\,2)(3\,4)$	$\mathfrak{b}(1)=\mathfrak{b}(2)\neq\mathfrak{b}(3)=\mathfrak{b}(4)$	$\langle 1,2\rangle\cap\langle 3,4\rangle=\emptyset$
$(1\,3)(2\,4)$	$\mathfrak{b}(1)=\mathfrak{b}(3)\neq\mathfrak{b}(2)=\mathfrak{b}(4)$	$\langle 1,3\rangle\cap\langle 2,4\rangle=\emptyset$
$(1\,4)(2\,3)$	$\mathfrak{b}(1)=\mathfrak{b}(4)\neq\mathfrak{b}(2)=\mathfrak{b}(3)$	$\langle 1,4\rangle\cap\langle 2,3\rangle=\emptyset$
Id	$\mathfrak{b}(1)=\mathfrak{b}(2)=\mathfrak{b}(3)=\mathfrak{b}(4)$	$\big\|\big\langle\sigma(1),\sigma(2)\big\rangle\cap\big\langle\sigma(3),\sigma(4)\big\rangle\big\|=1$
		for some or for all $\sigma\in S_4$

Figure 5.2: The Klein type of a quadrilateral A is characterized by the bifurcation configuration and by intersections of lines. The abbreviations $i := i_A$ and $\mathfrak{b}(i) := \mathfrak{b}(i)_A$ are used for $i = 1, 2, 3, 4$.

then by 1) is $\langle 1,2\rangle\cap\langle 3,4\rangle\neq\emptyset$. Hence by 2) the intersection has only one vertex.

The implication "⇒" for 4) follows from part 2) because the segments between the bifurcations have positive lengths. Conversely, each of the inequalities implies by 1) that $\langle 1,2\rangle\cap\langle 3,4\rangle$ is non-empty, hence 2) finishes the claim. □

5.27 Corollary. *The characterizations of the Klein type as displayed in* Figure 5.2 *is true.*

Proof. In comparison to the classification of the Klein type with the bifurcation type (Figure 5.1) the cases of Corollary 5.10 correspond to the cases of intersection of lines by Corollary 5.26, 1) and 3). □

We write a last characterization of the Klein type of quadrilaterals $(1, 2, 3, 4)$ according to intersection properties of the lines $\langle 1,2\rangle$ and $\langle 3,4\rangle$ only. See Figure 5.4 for diagrams.

5.28 Corollary. *The properties claimed in* Figure 5.3 *are true.*

Proof. For the upper table, an affirmation in a rectangle is partitioned into the exclusive sub-cases that are stated in the underneath rectangles. Only two doubts can arise. The fact that $\langle 1,2\rangle\cap\langle 3,4\rangle\neq\emptyset$ implies $\langle 1,2\rangle\cap\langle 3,4\rangle = \big[\mathfrak{b}(1,3,4),\mathfrak{b}(2,3,4)\big]$ as a vertex set is the content of Theorem 4. If $\mathfrak{b}(1,3,4)\neq\mathfrak{b}(2,3,4)$ then the ordering of these bifurcations in the line $\langle 1,2\rangle$ is determined

<table>
<tr><td colspan="4">$(1, 2, 3, 4)$ is a quadrilateral</td></tr>
<tr><td rowspan="3">$\langle 1, 2\rangle \cap \langle 3, 4\rangle = \emptyset$</td><td colspan="3">$\langle 1, 2\rangle \cap \langle 3, 4\rangle \neq \emptyset$
$\Updownarrow$
$\langle 1, 2\rangle \cap \langle 3, 4\rangle = [x, y]$
as a vertex set for $x = \mathfrak{b}(1, 3, 4), y = \mathfrak{b}(2, 3, 4)$</td></tr>
<tr><td colspan="2">$x \neq y$
then $\langle 1, 2\rangle = \langle 1, x][x, y][y, 2\rangle$</td><td rowspan="2">$x = y$</td></tr>
<tr><td>$\langle 3, 4\rangle$
$= \langle 3, x][x, y][y, 4\rangle$</td><td>$\langle 3, 4\rangle$
$= \langle 3, y][y, x][x, 4\rangle$</td></tr>
<tr><td>`(i)`</td><td>`(ii)`</td><td>`(iii)`</td><td>`(iv)`</td></tr>
</table>

<table>
<tr><td>$\Updownarrow$</td><td>$\Updownarrow$</td><td>$\Updownarrow$</td><td>$\Updownarrow$</td></tr>
<tr><td>$(1\,2)(3\,4)$</td><td>$(1\,3)(2\,4)$</td><td>$(1\,4)(2\,3)$</td><td>Id</td></tr>
</table>

Figure 5.3: The top table involves only conditions on the two lines $\langle 1, 2\rangle$, $\langle 3, 4\rangle$ and on their intersection. The affirmation in each rectangle is partitioned into the exclusive sub-cases that are stated in the underneath rectangles. The bottom table displays the cases of Klein type of quadrilaterals. These cases are in the indicated correspondence to the cases of the top table.

by Proposition 5.8. So the four cases provide a partition of the set of all quadrilaterals.

We compare the cases of this table to the cases of Figure 5.2. Literally, the intersection condition on Klein type $(1\,2)(3\,4)$ equals `(i)` and the intersection condition on Klein type Id is equivalent to `(iv)`. So these cases cover the same set of quadrilaterals. For all other quadrilaterals A we take as in the figure $x = \mathfrak{b}(1, 3, 4)$ and $y = \mathfrak{b}(2, 3, 4)$. In case `(ii)` Proposition 5.8 shows that $x = \mathfrak{b}(1, 2, 3)$ so that $\mathrm{Kl}(A) = (1\,3)(2\,4)$. The proposition show also that case `(iii)` implies $\mathrm{Kl}(A) = (1\,4)(2\,3)$. $\square$

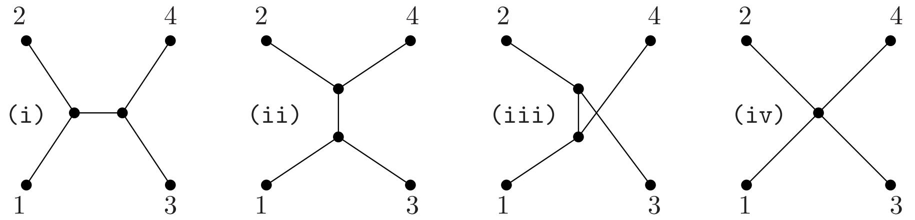


Figure 5.4: The four distinct cases of intersection between the two lines $\langle 1,2\rangle$ and $\langle 3,4\rangle$ of a quadrilateral $(1,2,3,4)$ according to Corollary 5.28.

5.5 Orientation and Concordance

5.29 Definition. When $A=(1,2,3,4)$ is a quadrilateral we introduce the *orientation* of the lines $\langle 1,2\rangle,\langle 3,4\rangle$ as

$$\vec{o}(1,2;3,4) := \begin{cases} 1 & \text{if } \operatorname{Kl}(1,2,3,4)=(1\,3)(2\,4),\\ -1 & \text{if } \operatorname{Kl}(1,2,3,4)=(1\,4)(2\,3),\\ 0 & \text{else.}\end{cases}$$

Sometimes we write $\vec{o}(A) := (1_A,2_A;3_A,4_A)$.

5.30 Note. For quadrilaterals $(1,2,3,4)$ holds by Coro. 5.28

$$\vec{o}(1,2;3,4) = \begin{cases} 1 & \text{if } \mathfrak{b}(1,3,4)\neq\mathfrak{b}(2,3,4) \text{ and } \langle 3,4\rangle\\ & =\langle 3,\mathfrak{b}(1,3,4)][\mathfrak{b}(1,3,4),\mathfrak{b}(2,3,4)][\mathfrak{b}(2,3,4),4\rangle,\\ -1 & \text{if } \mathfrak{b}(1,3,4)\neq\mathfrak{b}(2,3,4) \text{ and } \langle 3,4\rangle\\ & =\langle 3,\mathfrak{b}(2,3,4)][\mathfrak{b}(2,3,4),\mathfrak{b}(1,3,4)][\mathfrak{b}(1,3,4),4\rangle,\\ 0 & \text{if } \mathfrak{b}(1,3,4)=\mathfrak{b}(2,3,4).\end{cases}$$

5.31 Corollary. *For all quadrilaterals A and all isometries $h\in\operatorname{Is}(\mathcal{T})$ holds $\vec{o}\big(h(A)\big)=\vec{o}(A)$.*

Proof. The Klein type is invariant under isometries by Corollary 5.14. □

5.32 Proposition. *For quadrilaterals $A\in\mathcal{T}_{\mathrm{Q}}$ hold*

- $\vec{o}\big(\sigma(A)\big)=\vec{o}(A)$ *for all σ of the Klein 4-group $V<S_4$,*
- $\vec{o}\big(\sigma(A)\big)=-\vec{o}(A)$ *for $\sigma=(1\,2)$ or $\sigma=(3\,4)\in S_4$,*
- $\vec{o}(A)+\vec{o}\big(\sigma(A)\big)+\vec{o}\big(\sigma^2(A)\big)=0$ *for all cycles σ of length three.*

Proof. Recall from Corollary 5.17 that the Klein type is invariant under permutations of the Klein 4-group whence the first assertion is clear. Theorem 3 states further that $\mathrm{Kl}\big(\sigma(A)\big) = \sigma \mathrm{Kl}(A)\sigma^{-1}$. Thus

$$\begin{array}{rcccl} (1\,2)\,\big((1\,3)(2\,4)\big)\,(1\,2) &=& (3\,4)\,\big((1\,3)(2\,4)\big)\,(3\,4) &=& (1\,4)(2\,3), \\ (1\,2)\,\big((1\,4)(2\,3)\big)\,(1\,2) &=& (3\,4)\,\big((1\,4)(2\,3)\big)\,(3\,4) &=& (1\,3)(2\,4), \\ (1\,2)\,\big(\mathrm{Id}\big)\,(1\,2) &=& (3\,4)\,\big(\mathrm{Id}\big)\,(3\,4) &=& \mathrm{Id}, \\ \text{and} \quad (1\,2)\,\big((1\,2)(3\,4)\big)\,(1\,2) &=& (3\,4)\,\big((1\,2)(3\,4)\big)\,(3\,4) &=& (1\,2)(3\,4) \end{array}$$

prove the second statement.

First we prove the third statement for the permutation $(2\,3\,4)$. If A is not centered, then the set $\big\{A, \sigma(A), \sigma^2(A)\big\}$ has by Lemma 5.18 a quadrilateral of each of the three non-centered types. Thus the desired equation holds. If A is centered then $\sigma(A)$ and $\sigma^2(A)$ are centered, so all three terms in the sum vanish.

To extend the equation from $\sigma = (2\,3\,4)$ to all cycles of length three, note first, that the non-trivial Klein 4-group elements produce

$$\begin{array}{rcl} (1\,2)(3\,4)(2\,3\,4)(1\,2)(3\,4) &=& (1\,4\,3), \\ (1\,3)(2\,4)(2\,3\,4)(1\,3)(2\,4) &=& (4\,1\,2), \\ \text{and} \quad (1\,4)(2\,3)(2\,3\,4)(1\,4)(2\,3) &=& (3\,2\,1). \end{array}$$

So the desired identity holds for the cycles on the right-hand side, too: say $\sigma = (2\,3\,4)$ and $\tau \in V$ then

$$\vec{o}\big(A\big) + \vec{o}\big(\tau\sigma\tau(A)\big) + \vec{o}\big(\tau\sigma^2\tau(A)\big) = \vec{o}\big(\tau(A)\big) + \vec{o}\big(\sigma\tau(A)\big) + \vec{o}\big(\sigma^2\tau(A)\big) = 0.$$

The remaining four cycles of length three are inverse to the four permutations presented. Since for a cycle σ of length three holds $\sigma^{-1} = \sigma^2$ and $\sigma^{-2} = \sigma$, the proof is finished. □

5.33 Proposition. *If $A \in \mathcal{T}_{\mathrm{Q}}$ is a quadrilateral, $\langle 1_A, 2_A\rangle \cap \langle 3_A, 4_A\rangle \neq \emptyset$ and $B := \big(\mathfrak{b}(1_A, 3_A, 4_A), \mathfrak{b}(2_A, 3_A, 4_A), 3_A, 4_A\big)$ then $\mathfrak{b}(i)_B = \mathfrak{b}(i)_A$ for $i = 1, 2, 3, 4$.*

Proof. By Theorem 4 holds $[\mathfrak{b}(2)_A, \mathfrak{b}(1)_A] \subset \langle 1_A, 2_A\rangle$. Since $\mathfrak{b}(2)_A \in \langle 1_A, 3_A\rangle$ and $\mathfrak{b}(1)_A \in \langle 2_A, 3_A\rangle$ one has $\big(\mathfrak{b}(2)_A, \mathfrak{b}(1)_A, 3_A\big) < (1_A, 2_A, 3_A)$. Similarly $\big(\mathfrak{b}(2)_A, \mathfrak{b}(1)_A, 4_A\big) < (1_A, 2_A, 4_A)$. Thus by Proposition 4.6 hold

$$\begin{array}{rcccl} \mathfrak{b}(1_B, 2_B, 4_B) &=& \mathfrak{b}\big(\mathfrak{b}(2)_A, \mathfrak{b}(1)_A, 4_A\big) &=& \mathfrak{b}(1_A, 2_A, 4_A) \\ \text{and} \quad \mathfrak{b}(1_B, 2_B, 3_B) &=& \mathfrak{b}\big(\mathfrak{b}(2)_A, \mathfrak{b}(1)_A, 3_A\big) &=& \mathfrak{b}(1_A, 2_A, 3_A). \end{array}$$

The projective property of bifurcations mentioned in the same proposition gives

$$\begin{array}{lrcccl} & \mathfrak{b}(1_B,3_B,4_B) & = & \mathfrak{b}\big(\mathfrak{b}(1_A,3_A,4_A),3_A,4_A\big) & = & \mathfrak{b}(1_A,3_A,4_A) \\ \text{and} & \mathfrak{b}(2_B,3_B,4_B) & = & \mathfrak{b}\big(\mathfrak{b}(2_A,3_A,4_A),3_A,4_A\big) & = & \mathfrak{b}(2_A,3_A,4_A). \end{array}$$

□

5.34 Corollary. *If* $A \in \mathcal{T}_\mathrm{Q}$ *and* $B := \big(\mathfrak{b}(1_A,3_A,4_A),\mathfrak{b}(2_A,3_A,4_A),3_A,4_A\big)$ *then* $\vec{o}(B) = \vec{o}(A)$.

Proof. In the case that $\langle 1_A,2_A\rangle \cap \langle 3_A,4_A\rangle \neq \emptyset$ this follows from Proposition 5.33 because both quadrilaterals A and B have the same bifurcations, hence the same type. If $\langle 1_A,2_A\rangle \cap \langle 3_A,4_A\rangle = \emptyset$ then A is of type $(1\,2)(3\,4)$ by Corollary 5.27. In particular $1_B = \mathfrak{b}(1_A,3_A,4_A) = \mathfrak{b}(2_A,3_A,4_A) = 2_B$. Therefore $\mathfrak{b}(1)_B = \mathfrak{b}(2_B,3_B,4_B) = \mathfrak{b}(1_B,3_B,4_B) = \mathfrak{b}(2)_B$ whence by the same corollary B is of type Id or of type $(1\,2)(3\,4)$. □

5.35 Definition (Concordance)**.** For each quadrilateral $A \in \mathcal{T}_\mathrm{Q}$ the *concordance* of the lines $\langle 1_A,2_A\rangle$ and $\langle 3_A,4_A\rangle$ is given by the integer number

$$[1_A,2_A,3_A,4_A] := \vec{o}(A)\cdot \mathtt{diam}(A).$$

Sometimes we write $[A] := [1_A,2_A,3_A,4_A]$.

5.36 Note. By Coro. 5.28 the (orientation and hence the) concordance of two lines $\langle 1_A,2_A\rangle$ and $\langle 3_A,4_A\rangle$ is zero if their intersection $[x,y]$ has at most one vertex. Otherwise the concordance equals the length of $[x,y]$ in magnitude. It has positive sign if $[x,y]$ appears in $\langle 1_A,2_A\rangle$ in the same orientation as in $\langle 3_A,4_A\rangle$. If $[x,y]$ appears in $\langle 1_A,2_A\rangle$ in the opposite orientation as in $\langle 3_A,4_A\rangle$ then the concordance is negative. See also Note 5.30.

5.37 Proposition. *For all quadrilaterals* $A \in \mathcal{T}_\mathrm{Q}$ *and all isometries* $h \in \mathrm{Is}\,(\mathcal{T})$ *holds* $\big[h(A)\big] = [A]$.

Proof. The orientation is invariant under isometries by Corollary 5.31, the diameter is invariant under isometries by Corollary 5.23. □

5.38 Proposition. *For all quadrilaterals* $A \in \mathcal{T}_\mathrm{Q}$ *hold*

- $\big[\sigma(A)\big] = \big[A\big]$ *for all* σ *of the Klein 4-group* $V < S_4$,

- $\big[\sigma(A)\big] = -\big[A\big]$ *for* $\sigma = (1\,2)$ *or* $\sigma = (3\,4) \in S_4$,
- $\big[A\big] + \big[\sigma(A)\big] + \big[\sigma^2(A)\big] = 0$ *for all cycles* σ *of length three.*

Proof. By Corollary 5.24 one has $\mathtt{diam}\big(\sigma(A)\big) = \mathtt{diam}(A)$ for all permutations $\sigma \in S_4$. Proposition 5.32 proves the desired relations. □

5.39 Lemma. $[A] = \vec{o}(A) \cdot \mathrm{d}\big(\mathfrak{b}(1_A, 3_A, 4_A), \mathfrak{b}(2_A, 3_A, 4_A)\big)$.

Proof. It is only to check that $\mathtt{diam}(A) = \mathrm{d}\big(\mathfrak{b}(1,3,4), \mathfrak{b}(2,3,4)\big)$ in the case that A is of type $(1\,3)(2\,4)$ or of type $(1\,4)(2\,3)$. In these cases one has $\mathfrak{b}(1,3,4) \neq \mathfrak{b}(2,3,4)$ by Corollary 5.27. Thus Proposition 5.22 shows the equality. □

For vertices x, y lying on a line $\langle 3,4\rangle$ the concordance takes a simpler form.

5.40 Lemma. *If* $\langle 3,4\rangle$ *is a line and* $x, y \in \langle 3,4\rangle$ *are vertices then* $[x,y,3,4] = \vec{o}(x,y,3,4) \cdot \mathrm{d}(x,y)$.

Proof. If $x, y \in \langle 3,4\rangle$ then $\mathfrak{b}(x,3,4) = x$ and $\mathfrak{b}(y,3,4) = y$. The proof is finished by Lemma 5.39. □

5.41 Note. In comparison to Note 5.30 one has for vertices x, y on a line $\langle 3,4\rangle$

$$\vec{o}(x,y;3,4) \quad = \quad \begin{cases} 1 & \text{if} \quad x \neq y \quad \text{and} \quad \langle 3,4\rangle = \langle 3,x][x,y][y,4\rangle, \\ -1 & \text{if} \quad x \neq y \quad \text{and} \quad \langle 3,4\rangle = \langle 3,y][y,x][x,4\rangle, \\ 0 & \text{if} \quad x = y. \end{cases}$$

5.42 Lemma. *If* $\langle 3,4\rangle$ *is a line then for each geodesic* g *representing* $\langle 3,4\rangle$ *one has* $\big[g(k), g(l), 3, 4\big] = l - k$ *for all* k, l.

Proof. By Lemma 5.40 holds $\big[g(k), g(l), 3, 4\big] = \vec{o}\big(g(k), g(l), 3, 4\big) \cdot |l - k|$. If $l = k$ both formulas give value zero. If $k < l$ then $\langle 3,4\rangle = \langle 3, g(k)][g(k), g(l)][g(l), 4\rangle$ thus $\vec{o}\big(g(k), g(l), 3, 4\big) = 1$ and $l - k > 0$. In the case that $l < k$ one has $\langle 3,4\rangle = \langle 3, g(l)][g(l), g(k)][g(k), 4\rangle$ thus $\vec{o}\big(g(k), g(l), 3, 4\big) = -1$ and $l - k < 0$. □

5.43 Proposition. *For every* $A \in \mathcal{T}_{\mathrm{Q}}$ *holds* $\big[\mathfrak{b}(1,3,4)_A, \mathfrak{b}(2,3,4)_A, 3_A, 4_A\big] = [A]$.

Proof. The orientations of $B = \big(\mathfrak{b}(1,3,4)_A, \mathfrak{b}(2,3,4)_A, 3_A, 4_A\big)$ and of A are the same by Corollary 5.34. If $\langle 1_A, 2_A\rangle \cap \langle 3_A, 4_A\rangle \neq \emptyset$ then B and A have the same bifurcations by Proposition 5.33 thus they have the same diameter.

If $\langle 1_A, 2_A\rangle \cap \langle 3_A, 4_A\rangle = \emptyset$ then A has type $(1\,2)(3\,4)$ by Corollary 5.27. Then $\vec{o}(B) = \vec{o}(A) = 0$ proves $[B] = [A]$. $\square$

5.44 Corollary. *For every* $A = (1,2,3,4) \in \mathcal{T}_{\mathrm{Q}}$ *hold* $\big[1, \mathfrak{b}(2,3,4), 3, 4\big] = [A]$ *and* $\big[\mathfrak{b}(1,3,4), 2, 3, 4\big] = [A]$.

Proof. Note that $\big[1, \mathfrak{b}(2,3,4), 3, 4\big] \overset{\text{Prop. 5.43}}{=} \big[\mathfrak{b}(1,3,4), \mathfrak{b}\big(\mathfrak{b}(2,3,4),3,4\big), 3, 4\big] \overset{\text{Prop. 4.6}}{=} \big[\mathfrak{b}(1,3,4), \mathfrak{b}(2,3,4), 3, 4\big] \overset{\text{Prop. 5.43}}{=} [1,2,3,4]$. Hence $\big[\mathfrak{b}(1,3,4), 2, 3, 4\big] \overset{\text{Prop. 5.38}}{=} -\big[2, \mathfrak{b}(1,3,4), 3, 4\big] \overset{\text{Prop. 5.38}}{=} -[2,1,3,4] = [1,2,3,4]$. $\square$

5.45 Proposition. *For quadrilaterals* $A \in \mathcal{T}_{\mathrm{Q}}$ *holds* $[A] = \mathrm{d}\big(\mathfrak{b}(1)_A, \mathfrak{b}(4)_A\big) - \mathrm{d}\big(\mathfrak{b}(1)_A, \mathfrak{b}(3)_A\big)$.

Proof. By Proposition 5.22 holds $\mathtt{diam}(A) = \mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big)$ if and only if $\mathfrak{b}(i) \neq \mathfrak{b}(j)$. We use the Klein type characterization from Corollary 5.27. If A is of type $(1\,2)(3\,4)$ then $\mathfrak{b}(4)_A \neq \mathfrak{b}(1)_A \neq \mathfrak{b}(3)_A$. Thus $\mathrm{d}\big(\mathfrak{b}(1)_A, \mathfrak{b}(4)_A\big) - \mathrm{d}\big(\mathfrak{b}(1)_A, \mathfrak{b}(3)_A\big) = 0 = [A]$. If A is of type $(1\,3)(2\,4)$ then $\mathfrak{b}(4)_A \neq \mathfrak{b}(1)_A = \mathfrak{b}(3)_A$ thus $\mathrm{d}\big(\mathfrak{b}(1)_A, \mathfrak{b}(4)_A\big) - \mathrm{d}\big(\mathfrak{b}(1)_A, \mathfrak{b}(3)_A\big) = \mathtt{diam}(A) = [A]$. If A is of type $(1\,4)(2\,3)$ then $\mathfrak{b}(4)_A = \mathfrak{b}(1)_A \neq \mathfrak{b}(3)_A$ thus $\mathrm{d}\big(\mathfrak{b}(1)_A, \mathfrak{b}(4)_A\big) - \mathrm{d}\big(\mathfrak{b}(1)_A, \mathfrak{b}(3)_A\big) = -\mathtt{diam}(A) = [A]$. If A is centered then any distance between bifurcations is zero, thus the equality is complete. $\square$

Recall from Chapter 4 the distance of a vertex x to a line $l = \langle \mathbf{y}, \mathbf{z}\rangle$

$$\mathrm{d}(x, l) := \min_{y \in l} \big\{\mathrm{d}(x, y)\big\}.$$

5.46 Corollary. *For vertices* $x \in \mathrm{V}(\mathcal{T})$ *and lines* $\langle \mathbf{y}, \mathbf{z}\rangle$ *holds* $\mathrm{d}\big(x, \langle \mathbf{y}, \mathbf{z}\rangle\big) = \mathrm{d}\big(x, \mathfrak{b}(x, \mathbf{y}, \mathbf{z})\big) = [x, \mathbf{y}, x, \mathbf{z}]$.

Proof. $[x, \mathbf{y}, x, \mathbf{z}] \overset{\text{Prop. 5.45}}{=} \mathrm{d}\big(\mathfrak{b}(\mathbf{y}, x, \mathbf{z}), \mathfrak{b}(x, \mathbf{y}, x)\big) - \mathrm{d}\big(\mathfrak{b}(\mathbf{y}, x, \mathbf{z}), \mathfrak{b}(x, \mathbf{y}, \mathbf{z})\big) = \mathrm{d}\big(x, \mathfrak{b}(x, \mathbf{y}, \mathbf{z})\big) \overset{\text{Lemma 4.10}}{=} \mathrm{d}\big(x, \langle \mathbf{y}, \mathbf{z}\rangle\big)$. $\square$

Chapter 6

Pentagons

A *pentagon* P in a tree $\mathcal{T}$ is an ordered 5-tuple $(1_P, 2_P, 3_P, 4_P, 5_P)$ of symbols $1_P, 2_P, 3_P, 4_P, 5_P \in \mathcal{T} \cup \mathcal{T}(\infty)$ such that no two of theses symbols are coinciding boundary points.

Theorem 5 (Pentagon Equation)**.** *For all pentagons P in a tree $\mathcal{T}$ holds* $[1_P, 2_P, 4_P, 5_P] + [2_P, 3_P, 4_P, 5_P] + [3_P, 1_P, 4_P, 5_P] = 0.$

Proof. The equation shall first be shown in the special case that $1_P, 2_P$ and 3_P are vertices that lie on $\langle 4_P, 5_P \rangle$. We abbreviate $P = (x, y, z, 4, 5)$. So $x, y, z \in \langle 4, 5 \rangle$. Observe that

$$\begin{aligned} & [x, y, 4, 5] + [y, z, 4, 5] + [z, x, 4, 5] = 0 \\ \text{implies} \quad & \big[\sigma(x), \sigma(y), 4, 5\big] + \big[\sigma(y), \sigma(z), 4, 5\big] + \big[\sigma(z), \sigma(x), 4, 5\big] = 0 \end{aligned}$$

for all $\sigma \in S_3$. The group S_3 is generated by the cycles $(x\,y\,z)$ and $(x\,y)$. The permutation $\sigma = (x\,y\,z)$ plugged into the left-hand side of the second equation above reads $[y, z, 4, 5] + [z, x, 4, 5] + [x, y, 4, 5]$. This sum is a reordering of the left-hand side of the top equation hence is zero. The permutation $\sigma = (x\,y)$ used in the same way gives $[y, x, 4, 5] + [x, z, 4, 5] + [z, y, 4, 5]$. We apply Proposition 5.38 to get $[y, x, 4, 5] + [x, z, 4, 5] + [z, y, 4, 5] = -[x, y, 4, 5] - [z, x, 4, 5] - [y, z, 4, 5] = 0$.

By Lemma 5.40 holds

$$\begin{aligned} & [x, y, 4, 5] + [y, z, 4, 5] + [z, x, 4, 5] \\ = \quad & \vec{o}(x, y, 4, 5) \cdot \mathrm{d}(x, y) + \vec{o}(y, z, 4, 5) \cdot \mathrm{d}(y, z) + \vec{o}(z, x, 4, 5) \cdot \mathrm{d}(z, x). \end{aligned} \tag{6.1}$$

If $x = y$ the result comes directly from Proposition 5.38. We can assume that x, y, z are pairwise distinct. We can also assume that the vertices obey a special ordering, say

$$\langle 4, 5\rangle = \langle 4, x][x, y][y, z][z, 5\rangle.$$

Then $\vec{o}(x, y, 4, 5) = \vec{o}(y, z, 4, 5) = +1$ and $\vec{o}(z, x, 4, 5) = -1$. Thus Equation (6.1) becomes $\mathrm{d}(x, y) + \mathrm{d}(y, z) - \mathrm{d}(z, x) = 0$. This completes the proof of the Pentagon Equation for the case $1_P, 2_P, 3_P \in \langle 4_P, 5_P\rangle$.

For a general pentagon P it is $\mathfrak{b}(1_P, 4_P, 5_P), \mathfrak{b}(2_P, 4_P, 5_P), \mathfrak{b}(3_P, 4_P, 5_P) \in \langle 4_P, 5_P\rangle$ and therefore

$$\begin{aligned}
&[1_P, 2_P, 4_P, 5_P] + [2_P, 3_P, 4_P, 5_P] \\
\overset{\text{Prop. 5.43}}{=}\ & \big[\mathfrak{b}(1_P, 4_P, 5_P), \mathfrak{b}(2_P, 4_P, 5_P), 4_P, 5_P\big] \\
& \qquad + \big[\mathfrak{b}(2_P, 4_P, 5_P), \mathfrak{b}(3_P, 4_P, 5_P), 4_P, 5_P\big] \\
=\ & -\big[\mathfrak{b}(3_P, 4_P, 5_P), \mathfrak{b}(1_P, 4_P, 5_P), 4_P, 5_P\big] \\
\overset{\text{Prop. 5.43}}{=}\ & -[3_P, 1_P, 4_P, 5_P].
\end{aligned}$$

□

Chapter 7

Horocycles

Besides lines, triangles, quadrilaterals and pentagons, horocycles are important geometrical objects of a tree. We start by introducing the *horocycle distance*[1] *from* a vertex x *to* a vertex y *with respect to* a boundary point η.

$$B_\eta(x,y) := \mathrm{d}(y, \mathfrak{b}(x,y,\eta)) - \mathrm{d}(x, \mathfrak{b}(x,y,\eta)) \tag{7.1}$$

The vertex $\mathfrak{b}(x,y,\eta)$ is the bifurcation of the triangle (x,y,η) introduced in Chapter 4. Notice that by Corollary 5.46 holds

$$B_\eta(x,y) = [y,x,y,\eta] - [x,y,x,\eta] \tag{7.2}$$

for all vertices $x,y \in \mathrm{V}(\mathcal{T})$ and boundary points $\eta \in \mathcal{T}(\infty)$.

7.1 Proposition. *For all boundary points $\eta \in \mathcal{T}(\infty)$, vertices $x,y,z \in \mathrm{V}\mathcal{T}$ and isometries $h \in \mathrm{Is}\,(\mathcal{T})$ hold the relations*

$$\begin{aligned} B_\eta(x,x) &= 0, \\ B_\eta(x,y) &= -B_\eta(y,x), \\ B_\eta(x,y) + B_\eta(y,z) &= B_\eta(x,z), \\ \textit{and} \qquad B_{h(\eta)}\big(h(x),h(y)\big) &= B_\eta(x,y). \end{aligned} \tag{7.3}$$

Proof. Immediately from the definition, one has $B_\eta(x,x) = 0$ and $B_\eta(x,y) = -B_\eta(y,x)$ for all vertices x,y and boundary points η. For a proof of the remaining two rules we use the expression from Eqn (7.2). Since the concordance $[\cdot]$ is

[1] The horocycle distance defined here has opposite sign compared to [Wei04] in order to adopt to the nomenclature used in [Bal95].

invariant under isometries by Prop. 5.37 the bifurcation distance is invariant, too. To verify the sum rule we calculate

$$\begin{aligned} B_\eta(x,y) + B_\eta(y,z) &= [y,x,y,\eta] - [x,y,x,\eta] + [z,y,z,\eta] - [y,z,y,\eta] \\ &\overset{\text{Theorem 5}}{=} -[x,z,y,\eta] - [x,y,x,\eta] + [z,y,z,\eta] \\ &\overset{\text{Prop. 5.38}}{=} [z,y,x,\eta] + [y,x,z,\eta] - [x,y,x,\eta] + [z,y,z,\eta] \\ &\overset{\text{Theorem 5}}{=} [z,x,z,\eta] - [x,z,x,\eta] = B_\eta(x,z). \end{aligned}$$

□

7.2 Definition (Horocycles)**.** For boundary points η we introduce a relation on the set of vertices of the tree $\mathcal{T}$:

$$x \sim_\eta y \quad :\Longleftrightarrow \quad B_\eta(x,y) = 0.$$

It follows directly form the sum rule for the horocycle distance (7.3), that this relation is an equivalence relation. The classes are called *horocycles centered* at $\eta \in \mathcal{T}(\infty)$.

The vertices in Figure 7.1 are labeled by the horocycle distance $B_\omega(x,\cdot)$ from the vertex x. Vertices sharing the same label lie on a horocycle: since $B_\omega(y,z) = B_\omega(y,x) + B_\omega(x,z) = -B_\omega(x,y) + B_\omega(x,z)$, it is clear that the distance $B_\omega(y,z)$ is zero if and only if $B_\omega(x,y) = B_\omega(x,z)$, i.e. if y and z have the same label. A horocycle centered at ω may be thought of as the location of a wave front that propagates from ω with constant velocity over the tree.

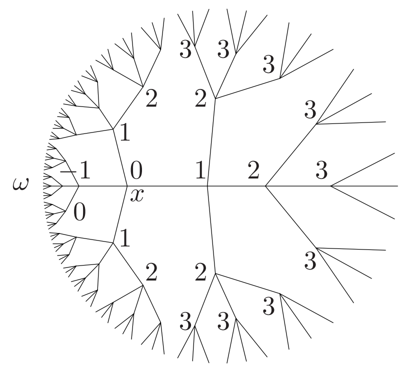


Figure 7.1: Horocycles centered at $\omega \in \mathcal{T}(\infty)$.

For vertices that lie on a line to or from a boundary point γ, their γ-horocycle distance can be expressed more explicitly. We recall the orientation $\vec{o}$ of two lines from Section 5.4. If $\eta \neq \xi$ are boundary points and two vertices x, y lie on the line (η, ξ) then obviously $\mathfrak{b}(x, \eta, \xi) = x$ and $\mathfrak{b}(y, \eta, \xi) = y$. Thus in this restriction $\vec{o}$ simplifies to

$$\vec{o}(x, y; \eta, \xi) \quad = \quad \begin{cases} 1 & \text{if} \quad x \neq y \quad \text{and} \quad (\eta, \xi) = (\eta, x][x, y][y, \xi), \\ -1 & \text{if} \quad x \neq y \quad \text{and} \quad (\eta, \xi) = (\eta, y][y, x][x, \xi), \\ 0 & \text{if} \quad x = y. \end{cases}$$

7.3 Lemma. *If (η, ξ) is a bi-infinite line and two vertices x, y lie on (η, ξ) then*

$$\begin{aligned} B_\eta\big(x, y\big) &= \vec{o}(x, y; \eta, \xi) \cdot d(x, y) = [x, y, \eta, \xi] \\ \textit{and} \quad B_\xi\big(x, y\big) &= -B_\eta\big(x, y\big). \end{aligned}$$

If a bi-infinite geodesic g has past $\alpha(g) = \eta$ then for all integers k, l holds

$$B_\eta\big(g(k), g(l)\big) = l - k.$$

If a bi-infinite geodesic g has future $\omega(g) = \xi$ then for all integers k, l holds

$$B_\xi\big(g(k), g(l)\big) = k - l.$$

Proof. Recall that $B_\eta(x, y) = \mathrm{d}\big(y, \mathfrak{b}(x, y, \eta)\big) - \mathrm{d}\big(x, \mathfrak{b}(x, y, \eta)\big)$. If $x \in (\eta, y]$, then $x = \mathfrak{b}(x, y, \eta)$. Hence $B_\eta(x, y) = \mathrm{d}(y, x) - \mathrm{d}(x, x) = \mathrm{d}(x, y)$. In case that $y \in (\eta, x]$ one has $y = \mathfrak{b}(x, y, \eta)$ and obtains $B_\eta(x, y) = \mathrm{d}(y, y) - \mathrm{d}(x, y) = -\mathrm{d}(x, y)$. The concordance $[x, y, \eta, \xi]$ equals $\vec{o}(x, y; \eta, \xi) \cdot \mathrm{d}(x, y)$ by Lemma 5.40. The second statement follows directly from Proposition 5.38. The remaining two statements are proved by Lemma 5.42. □

7.4 Proposition. *Assume γ is a boundary point. Every bi-infinite line l with past or future γ intersects each horocycle centered at γ in exactly one vertex.*

Proof. We fix an γ-horocycle H by $x \in \mathrm{V}(\mathcal{T})$ through $H = \big\{y \in \mathrm{V}(\mathcal{T}) : B_\gamma(x, y) = 0\big\}$. We choose a bi-infinite geodesic g with past γ and put $m := B_\gamma\big(x, g(0)\big) \in \mathbb{Z}$. Now by Lemma 7.3 one has for all $n \in \mathbb{Z}$

$$\begin{aligned} B_\gamma\big(x, g(n)\big) \quad &= \quad B_\gamma\big(x, g(0)\big) + B_\gamma\big(g(0), g(n)\big) \\ = \quad m + (n - 0) \quad &= \quad m + n. \end{aligned}$$

This shows $H \cap g = \{g(-m)\}$. If a line l has future γ, then $l = (\alpha, \gamma)$ for some $\alpha \in \mathcal{T}(\infty)$. The line (γ, α) has the same vertices as l and has past γ. So this case reduces to the first one. $\square$

Chapter 8

The Geodesic Space of a Tree

The *geodesic space* $\mathcal{G}$ of a tree $\mathcal{T}$ consists by definition of all bi-infinite geodesics. These are bi-infinite paths $g : \mathcal{T}_2 \to \mathcal{T}$ without backtracking, which means that $g(i+2) \neq g(i)$ for all $i \in \mathbb{Z}$. Elements of $\mathcal{G}$ shall be called *geodesics* for simplicity.

8.1 The Coordinate Space

Theorem 1 suggests a parameterization of $\mathcal{G}$ by assigning to any geodesic g the velocity $\mathcal{V}(g) \in \mathcal{T}(\infty) \times \mathcal{T}(\infty)$ and an integer number. It was shown there that there is an identification of $\mathcal{G}$ with

$$\mathcal{C} \quad := \quad \Big(\big(\mathcal{T}(\infty) \times \mathcal{T}(\infty) \big) - \Delta \Big) \times \mathbb{Z}$$

for $\Delta = \{(\gamma, \gamma) : \gamma \in \mathcal{T}(\infty)\}$. The set $\mathcal{C}$ is called *coordinate space* of $\mathcal{G}$. There is an invertible shift operator L on $\mathcal{G}$ defined for geodesics g by

$$\mathrm{L}(g)(i) := g(i+1).$$

for all $i \in \mathbb{Z}$. A shift along a geodesic g does not change the velocity $\mathcal{V}(g) = \big(\alpha(g), \omega(g)\big)$. When a third coordinate shall be assigned to g such that the shift operator L acts on geodesics by increasing this number by one, then there is still a degree of freedom consisting in a translation along g. A reference vertex will be chosen to fix coordinates. This construction is prepared now.

8.1 Definition (x-coordinates)**.** When a *base point* $x \in \mathcal{T}$ is chosen, we define a map

$$\mathcal{X}_x : \begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathbb{Z} \\ g & \longmapsto & [x, g(0), \alpha(g), \omega(g)]. \end{array}$$

The integer number $\mathcal{X}_x(g)$ is called the *x-position* of g. Images of the map

$$\kappa_x : \begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{C} \\ g & \longmapsto & \big(\mathcal{V}(g), \mathcal{X}_x(g)\big) \end{array}$$

are called *x-coordinates* of g. The x-coordinates of a geodesic g consist of the velocity $\mathcal{V}(g) = \big(\alpha(g), \omega(g)\big)$ from Eqn. (3.4) and the x-position $\mathcal{X}_x(g) = \big[x, g(0), \alpha(g), \omega(g)\big]$ of g that involves the concordance $[\cdot,\cdot,\cdot,\cdot]$ defined in Section 5.5.

Recall from Proposition 5.38 in Section 5.5 and Theorem 5 in Chapter 6 the following relations of the concordance $[\cdot,\cdot,\cdot,\cdot]$ for vertices x, y, z, distinct boundary points $\eta \neq \xi$ and isometries $h \in \mathrm{Is}\,(\mathcal{T})$:

$$\begin{array}{rcl} [y, x, \eta, \xi] & = & -[x, y, \eta, \xi], \\ {[x, y, \eta, \xi] + [y, z, \eta, \xi]} & = & [x, z, \eta, \xi], \\ {[x, y, \xi, \eta]} & = & -[x, y, \eta, \xi], \\ \text{and} \quad \big[h(x), h(y), h(\eta), h(\xi)\big] & = & [x, y, \eta, \xi]. \end{array} \tag{8.1}$$

There is an interpretation of the x-position $\mathcal{X}_x(g) = \big[x, g(0), \alpha(g), \omega(g)\big]$ as explained in Figure 8.1.

Theorem 6. *The map κ_x is a bijection from the geodesic space $\mathcal{G}$ to the coordinate space $\mathcal{C}$ for all base points $x \in \mathrm{V}\mathcal{T}$. The action of $\mathbb{Z}$ on the coordinate space defined as*

$$\widehat{\mathrm{L}}^k(\eta, \xi, n) := (\eta, \xi, n + k) \tag{8.2}$$

for $k \in \mathbb{Z}$ and $(\eta, \xi, n) \in \mathcal{C}$ satisfies $\kappa_x \circ \mathrm{L}^k = \widehat{\mathrm{L}}^k \circ \kappa_x$ for all base points $x \in \mathrm{V}\mathcal{T}$.

Proof. It was shown in Theorem 1 that for each pair $\eta \neq \xi$ of boundary points there exists a geodesic g with velocity $\mathcal{V}(g) = (\eta, \xi)$. It was shown also that

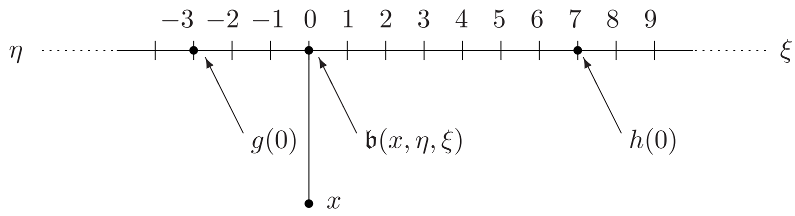


Figure 8.1: Two geodesics g, h with velocity $\mathcal{V}(g) = \mathcal{V}(h) = (\eta, \xi)$ obtain a third integer coordinate $\mathcal{X}_x(g)$ depending on a base point x. By Proposition 5.43 and Lemma 5.39 holds $\mathcal{X}_x(g) = \big[x, g(0), \eta, \xi\big] = \big[\mathfrak{b}(x,\eta,\xi), g(0), \eta, \xi\big] = \vec{o}\big(\mathfrak{b}(x,\eta,\xi), g(0), \eta, \xi\big) \cdot \mathrm{d}\big(\mathfrak{b}(x,\eta,\xi), g(0)\big)$. The orientation $\vec{o}\big(\mathfrak{b}(x,\eta,\xi), g(0), \eta, \xi\big)$ assumes the values $+1$, 0 or 1 depending on how the vertices $\mathfrak{b}(x,\eta,\xi)$ and $g(0)$ are ordered on the line (η, ξ) (compare Note 5.41). Here $\mathcal{X}_x(g) = -3$ and $\mathcal{X}_x(h) = 7$.

two geodesics g, h with the same velocity (η, ξ) are linked by the shift operator L through $h = \mathrm{L}^k(g)$ for some $k \in \mathbb{Z}$. Bijectivity of κ_x follows therefore from bijectivity of the map $k \mapsto \mathcal{X}\big(\mathrm{L}^k(g)\big)$ from $\mathbb{Z}$ to $\mathbb{Z}$ for all geodesics g:

$$\begin{aligned}
\mathcal{X}_x\big(\mathrm{L}^k(g)\big) &= \Big[x, \big(\mathrm{L}^k(g)\big)(0), \alpha\big(L^k(g)\big), \omega\big(\mathrm{L}^k(g)\big)\Big] \\
&= \big[x, g(k), \alpha(g), \omega(g)\big] \\
&\overset{(8.1)}{=} \big[x, g(0), \alpha(g), \omega(g)\big] + \big[g(0), g(k), \alpha(g), \omega(g)\big] \\
&= \mathcal{X}_x(g) + \big[g(0), g(k), \alpha(g), \omega(g)\big] \\
&\overset{\text{Lemma 5.42}}{=} \mathcal{X}_x(g) + k.
\end{aligned}$$

The result shows that the map κ_x is indeed a bijection. The action $\widehat{\mathrm{L}}^k(\eta, \xi, n) := (\eta, \xi, n + k)$ of $\mathbb{Z}$ on $\mathcal{C}$ satisfies $\widehat{\mathrm{L}} \circ \kappa_x = \kappa_x \circ \mathrm{L}$:

$$\begin{aligned}
\kappa_x \circ \mathrm{L}^k(g) &= \Big(\mathcal{V}\big(\mathrm{L}^k(g)\big), \mathcal{X}_x\big(\mathrm{L}^k(g)\big)\Big) = \big(\mathcal{V}(g), \mathcal{X}_x(g) + k\big) \\
&= \widehat{\mathrm{L}}^k\big(\mathcal{V}(g), \mathcal{X}_x(g)\big) = \widehat{\mathrm{L}}^k \circ \kappa_x(g).
\end{aligned}$$

□

For completeness, a formula expressing a change of the base point and a formula for the image of a geodesic in x-coordinates under isometries are given. These formulas will not be used.

8.2 Proposition. *For all base points x, y and all elements (η, ξ, n) of the coordinate space $\mathcal{C}$ holds $\kappa_y \circ \kappa_x^{-1}\big(\eta, \xi, n\big) = \big(\eta, \xi, n - [x, y, \eta, \xi]\big)$.*

Proof. The velocity of a geodesic does not depend on the base point, so it is the change in position that has to be calculated. We take two base points x, y.

$$\begin{aligned}\mathcal{X}_y(g) &= \big[y, g(0), \eta, \xi\big] \overset{(8.1)}{=} \big[y, x, \eta, \xi\big] + \big[x, g(0), \eta, \xi\big] \\ &= \big[y, x, \eta, \xi\big] + \mathcal{X}_x(g) \overset{(8.1)}{=} \mathcal{X}_x(g) - [x, y, \eta, \xi].\end{aligned}$$

□

8.3 Proposition. *For all base points x and all isometries $h \in \mathrm{Is}(\mathcal{T})$ the map $h_x : \mathcal{C} \to \mathcal{C}$ defined by $h_x(\eta, \xi, n) := \Big(h(\eta), h(\xi), n + \big[x, h(x), h(\eta), h(\xi)\big]\Big)$ satisfies $h_x \circ \kappa_x = \kappa_x \circ h$.*

Proof. If a geodesic g has velocity (η, ξ) and h is an isometry then $h(g)$ has velocity $\big(h(\eta), g(\xi)\big)$. The position of the geodesic $h(g)$ is given by

$$\begin{aligned}\mathcal{X}_x\big(h(g)\big) &= \big[x, h(g)(0), h(\eta), h(\xi)\big] \\ &\overset{(8.1)}{=} \big[x, h(x), h(\eta), h(\xi)\big] + \big[h(x), h(g)(0), h(\eta), h(\xi)\big] \\ &\overset{(8.1)}{=} \big[x, h(x), h(\eta), h(\xi)\big] + [x, g(0), \eta, \xi] \\ &= \big[x, h(x), h(\eta), h(\xi)\big] + \mathcal{X}_x(g).\end{aligned}$$

□

The following operator will help to simplify arguments in the next section and in Section 9.3.

8.4 Definition (Parity Operator)**.** There is a bijection $\mathrm{p} : \mathcal{G} \to \mathcal{G}$ defined for $g \in \mathcal{G}$ by $\mathrm{p}(g)(i) := g(-i)$ for all $i \in \mathbb{Z}$. The operator p is called the *parity operator*. Its action is a special inversion of the velocity of g.

8.5 Lemma. *The parity operator $\mathrm{p} : \mathcal{G} \to \mathcal{G}$ satisfies $\mathrm{p} \circ \mathrm{p} = \mathrm{Id}|_{\mathcal{G}}$.*

Proof. For every geodesic $g \in \mathcal{G}$ and all $i \in \mathbb{Z}$ holds $\mathrm{p}\Big(\mathrm{p}\big(g\big)\Big)(i) = \mathrm{p}\big(g\big)(-i) = g(i)$. □

8.6 Proposition. *The operator $\widehat{\mathrm{p}} : \mathcal{C} \to \mathcal{C}$ defined as $\widehat{\mathrm{p}}(\eta, \xi, n) := (\xi, \eta, -n)$ satisfies $\kappa_x \circ \mathrm{p} = \widehat{\mathrm{p}} \circ \kappa_x$ for all base points x.*

Proof. Let g be a geodesic in $\mathcal{G}$. Note that $\alpha\big(\mathrm{p}(g)\big) = \omega(g)$, $\omega\big(\mathrm{p}(g)\big) = \alpha(g)$ and $\mathrm{p}(g)(0) = g(0)$. Thus

$$\begin{aligned}
\widehat{\mathrm{p}} \circ \kappa_x(g) &= \widehat{\mathrm{p}}\big(\alpha(g), \omega(g), \mathcal{X}_x(g)\big) = \big(\omega(g), \alpha(g), -\mathcal{X}_x(g)\big) \\
&= \big(\omega(g), \alpha(g), -[x, g(0), \alpha(g), \omega(g)]\big) \\
&\overset{(8.1)}{=} \big(\omega(g), \alpha(g), [x, g(0), \omega(g), \alpha(g)]\big) \\
&= \Big(\alpha\big(\mathrm{p}(g)\big), \omega\big(\mathrm{p}(g)\big), \big[x, p(g)(0), \alpha\big(\mathrm{p}(g)\big), \omega\big(\mathrm{p}(g)\big)\big]\Big) \\
&= \kappa_x \circ \mathrm{p}(g).
\end{aligned}$$

□

8.7 Proposition. *One has for all $k \in \mathbb{Z}$ the equality* $\mathrm{p} \circ \mathrm{L}^k = \mathrm{L}^{-k} \circ \mathrm{p}$.

Proof. We choose a base point x, a geodesic $g \in \mathcal{G}$ and put $\alpha := \alpha(g)$, $\omega := \omega(g)$ and $n := \mathcal{X}_x(g)$. Then

$$\begin{aligned}
\kappa_x \circ \mathrm{p} \circ \mathrm{L}^k(g) &= \widehat{\mathrm{p}} \circ \widehat{\mathrm{L}}^k \circ \kappa_x(g) = \widehat{\mathrm{p}} \circ \widehat{\mathrm{L}}^k(\alpha, \omega, n) \\
&= \widehat{\mathrm{p}}(\alpha, \omega, n+k) = (\omega, \alpha, -n-k) = \widehat{\mathrm{L}}^{-k}(\omega, \alpha, -n) \\
&= \widehat{\mathrm{L}}^{-k} \circ \widehat{\mathrm{p}}(\alpha, \omega, n) = \widehat{\mathrm{L}}^{-k} \circ \widehat{\mathrm{p}} \circ \kappa_x(g) = \kappa_x \circ \mathrm{L}^{-k} \circ \mathrm{p}(g).
\end{aligned}$$

Since κ_x is a bijection, the claim follows. □

8.2 Unstable and Stable Manifolds

The *unstable manifold* $U(g)$ of a geodesic g is defined as

$$U(g) := \big\{h \in \mathcal{G} : \alpha(h) = \alpha(g)\big\}.$$

The *stable manifold* $S(g)$ of a geodesic g is given by

$$S(g) := \big\{h \in \mathcal{G} : \omega(h) = \omega(g)\big\}.$$

These definitions depend only on the past respectively the future of a geodesic. Thus the manifolds will also be defined by boundary points in the obvious way. One has $U(g) = U\big(\alpha(g)\big)$ and $S(g) = S\big(\omega(g)\big)$ for geodesics $g \in \mathcal{G}$. (See Figure 8.2.)

The *strongly unstable manifold* $U^S(g)$ of a geodesic g takes the form

$$U^S(g) := \Big\{h \in \mathcal{G} : \quad \alpha(h) = \alpha(g) \quad \text{and} \quad B_{\alpha(g)}\big(h(0), g(0)\big) = 0\Big\}.$$

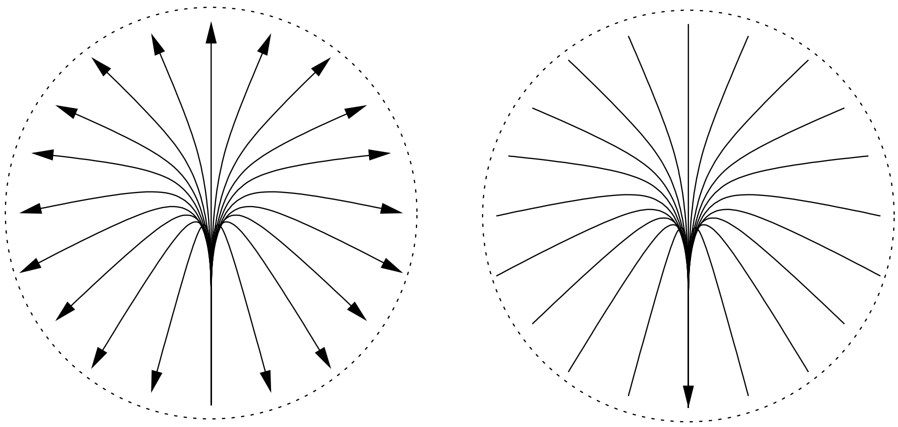


Figure 8.2: The unstable manifold of α consists of all geodesics that have α as their past (left). The stable manifold of ω consists of all geodesics that have ω as their future (right).

The *strongly stable manifold* $S^S(g)$ of a geodesic g is given by

$$S^S(g) := \Big\{ h \in \mathcal{G} : \quad \omega(h) = \omega(g) \quad \text{and} \quad B_{\omega(g)}\big(h(0), g(0)\big) = 0 \Big\}.$$

8.8 Proposition (New-Future and New-Past Maps)**.**
For each $\xi \in \mathcal{T}(\infty)$ there is a unique map called the new-future map to ξ

$$\mathcal{N}_\xi^{Fut} : \begin{array}{ccc} \mathcal{G} - U(\xi) & \longrightarrow & S(\xi) \\ g & \longmapsto & \mathcal{N}_\xi^{Fut}(g) \end{array} \tag{8.3}$$

such that $\mathcal{N}_\xi^{\mathrm{Fut}}(g) \in U^S(g)$ for each geodesic g in the domain of definition.
For each $\eta \in \mathcal{T}(\infty)$ there is a unique map called the new-past map to η

$$\mathcal{N}_\eta^{Pas} : \begin{array}{ccc} \mathcal{G} - S(\eta) & \longrightarrow & U(\eta) \\ g & \longmapsto & \mathcal{N}_\eta^{Pas}(g) \end{array} \tag{8.4}$$

such that $\mathcal{N}_\eta^{\mathrm{Pas}}(g) \in S^S(g)$ for each geodesic g in the domain of definition.

Proof. To establish the new-future map to a boundary point ξ it is to show that for every geodesic $g \in \mathcal{G}$ such that $\alpha(g) \neq \xi$ there is a unique geodesic h with velocity $\mathcal{V}(h) = \big(\alpha(g), \xi\big)$ such that $B_{\alpha(g)}\big(h(0), g(0)\big) = 0$.

It was shown in Theorem 1 that there are geodesics h with $\mathcal{V}(h) = (\alpha(g), \xi)$ and that two such geodesics are translates of each other by the shift operator L. One of these geodesics h shall be fixed. By Proposition 7.4 the geodesic h intersects the horocycle through $g(0)$ in exactly one vertex, say in $h(k)$. Thus $l = k$ is the unique solution for $B_\alpha\big(L^l(h)(0), g(0)\big) = 0$ because $B_\alpha\big(L^k(h)(0), g(0)\big) = B_\alpha\big(h(k), g(0)\big) = 0$.

As for the new-past map we choose a boundary point η and a geodesic g such that $\omega(g) \neq \eta$. Since $\alpha\big(\mathrm{p}(g)\big) = \omega(g) \neq \eta$ there is by the first part a unique geodesic h such that $\mathcal{V}(h) = \big(\omega(g), \eta\big)$ and $B_{\alpha(\mathrm{p}(g))}\big(h(0), \mathrm{p}(g)(0)\big) = 0$. These two conditions are equivalent to the conditions $\mathcal{V}\big(\mathrm{p}(h)\big) = \big(\eta, \omega(g)\big)$ and $B_{\omega(g)}\big(\mathrm{p}(h)(0), g(0)\big) = 0$. So $\mathrm{p}(h)$ is the unique geodesic with the two last-mentioned properties. □

8.9 Proposition. *The diagram*

$$\begin{array}{ccc} \mathcal{G} - U(\xi) & \xrightarrow{\mathcal{N}_\xi^{\mathrm{Fut}}} & \mathcal{G} \\ \mathrm{p}\big\downarrow & & \big\downarrow\mathrm{p} \\ \mathcal{G} - S(\xi) & \xrightarrow[\mathcal{N}_\xi^{\mathrm{Pas}}]{} & \mathcal{G} \end{array}$$

commutes for all boundary points ξ. That is $\mathrm{p} \circ \mathcal{N}_\xi^{\mathrm{Fut}} = \mathcal{N}_\xi^{\mathrm{Pas}} \circ \mathrm{p}$ for all boundary points ξ.

Proof. We fix a boundary point ξ and a geodesic g with velocity (α, ω) such that $\alpha \neq \xi$. First one obtains that $\mathcal{V}\big(\mathrm{p} \circ \mathcal{N}_\xi^{\mathrm{Fut}}(g)\big) = (\xi, \alpha) = \mathcal{V}\big(\mathcal{N}_\xi^{\mathrm{Pas}} \circ \mathrm{p}(g)\big)$ by Proposition 8.6. We set $h_1 := \mathrm{p} \circ \mathcal{N}_\xi^{\mathrm{Fut}}(g)$ and $h_2 := \mathcal{N}_\xi^{\mathrm{Pas}} \circ \mathrm{p}(g)$. Since both geodesics have the same velocity, it is sufficient to prove $h_1(0) = h_2(0)$. By Proposition 7.4 this follows from $B_\alpha\big(h_1(0), h_2(0)\big) = 0$, which can be verified.

$$\begin{aligned} & B_\alpha\Big(\mathrm{p} \circ \mathcal{N}_\xi^{\mathrm{Fut}}(g)(0), \mathcal{N}_\xi^{\mathrm{Pas}} \circ \mathrm{p}(g)(0)\Big) \\ \stackrel{(7.3)}{=} \quad & B_\alpha\Big(\mathrm{p}\big(\mathcal{N}_\xi^{\mathrm{Fut}}(g)\big)(0), \mathrm{p}(g)(0)\Big) + B_\alpha\Big(\mathrm{p}(g)(0), \mathcal{N}_\xi^{\mathrm{Pas}}\big(\mathrm{p}(g)\big)(0)\Big) \\ = \quad & B_\alpha\Big(\mathcal{N}_\xi^{\mathrm{Fut}}(g)(0), g(0)\Big) + B_\alpha\Big(\mathrm{p}(g)(0), \mathcal{N}_\xi^{\mathrm{Pas}}\big(\mathrm{p}(g)\big)(0)\Big) \\ = \quad & 0 + 0 \end{aligned}$$

by the definition of the new-future map $\mathcal{N}_\xi^{\mathrm{Fut}}$ (note that $\alpha(g) = \alpha$) and the definition of the new-past map $\mathcal{N}_\xi^{\mathrm{Pas}}$ (note that $\omega\big(\mathrm{p}(g)\big) = \alpha$). □

Theorem 7. *For all base points x and all boundary points ξ, the map $\mathcal{N}^{\text{Fut}}_{x,\xi} : \mathcal{C} - U(\xi) \to \mathcal{C}$ defined as*

$$\mathcal{N}^{Fut}_{x,\xi}(\alpha,\omega,n) := \begin{cases} \big(\alpha,\xi,n+[x,\alpha,\omega,\xi]\big) & \text{if} \quad \omega \neq \xi \\ (\alpha,\xi,n) & \text{if} \quad \omega = \xi \end{cases}$$

makes the diagram

$$\begin{array}{ccc} \mathcal{G} - U(\xi) & \xrightarrow{\kappa_x} & \mathcal{C} - U(\xi) \\ {\scriptstyle \mathcal{N}^{\text{Fut}}_{\xi}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{N}^{\text{Fut}}_{x,\xi}} \\ \mathcal{G} & \xrightarrow[\kappa_x]{} & \mathcal{C} \end{array}$$

commute. That is $\kappa_x \circ \mathcal{N}^{\text{Fut}}_{\xi} = \mathcal{N}^{\text{Fut}}_{x,\xi} \circ \kappa_x$ for all base points x and all boundary points ξ.

Proof. Let $g \in \mathcal{G} - U(\xi)$ and put $h := \mathcal{N}^{\text{Fut}}_{\xi}(g)$. These choices shall be written as $\kappa_x(g) = \big(\alpha,\omega,\mathcal{X}_x(g)\big)$ for $\alpha \neq \xi$ and $\kappa_x(h) = \big(\alpha,\xi,\mathcal{X}_x(h)\big)$ such that $B_\alpha\big(h(0),g(0)\big) = 0$. We are interested in the integer number $\mathcal{X}_x(h) - \mathcal{X}_x(g)$ which equals by definition $\big[x,h(k),\alpha,\xi\big] - \big[x,g(k),\alpha,\omega\big]$ for $k = 0$. We show in a first step that this expression is independent of $k \in \mathbb{Z}$.

$$\begin{aligned} &\big[x,h(k),\alpha,\xi\big] - \big[x,g(k),\alpha,\omega\big] - \big[x,h(0),\alpha,\xi\big] + \big[x,g(0),\alpha,\omega\big] \\ &\overset{(8.1)}{=} \quad \big[h(0),x,\alpha,\xi\big] + \big[x,h(k),\alpha,\xi\big] - \big[g(0),x,\alpha,\omega\big] - \big[x,g(k),\alpha,\omega\big] \\ &\overset{(8.1)}{=} \quad \big[h(0),h(k),\alpha,\xi\big] - \big[g(0),g(k),\alpha,\omega\big] \\ &\overset{\text{Lemma 5.42}}{=} \quad (k-0) - (k-0) \quad = \quad 0. \end{aligned}$$

Since $B_\alpha\big(h(0),g(0)\big) = 0$ it holds for all $k \in \mathbb{Z}$

$$\begin{aligned} B_\alpha\big(h(k),g(k)\big) &\overset{(7.3)}{=} B_\alpha\big(h(k),h(0)\big) + B_\alpha\big(h(0),g(0)\big) + B_\alpha\big(g(0),g(k)\big) \\ &= B_\alpha\big(h(k),h(0)\big) + B_\alpha\big(g(0),g(k)\big) \\ &\overset{\text{Lemma 7.3}}{=} (0-k) + (k-0) \quad = \quad 0. \end{aligned}$$

Take $k \in \mathbb{Z}$ such that $g(k) = \mathfrak{b}(\alpha,\omega,\xi) \in (\alpha,\xi)$. Since $h(k) \in (\alpha,\xi)$ by definition and since $B_\alpha\big(h(k),g(k)\big) = 0$ Proposition 7.4 shows that $h(k) = g(k)$. This allows to calculate

$$\begin{aligned} \mathcal{X}_x(h) - \mathcal{X}_x(g) &= \big[x,\mathfrak{b}(\alpha,\omega,\xi),\alpha,\xi\big] - \big[x,\mathfrak{b}(\alpha,\omega,\xi),\alpha,\omega\big] \\ &\overset{\text{Coro. 5.44}}{=} [x,\omega,\alpha,\xi] - [x,\xi,\alpha,\omega] \overset{\text{Prop. 5.38}}{=} -[x,\omega,\xi,\alpha] - [x,\xi,\alpha,\omega] \\ &\overset{\text{Prop. 5.38}}{=} [x,\alpha,\omega,\xi]. \end{aligned}$$

if (x,α,ω,ξ) is a quadrilateral. Otherwise $\omega = \xi$, so $h = g$ and therefore $\kappa_x(h) = \kappa_x(g)$. □

Theorem 8. *For all base points x and all boundary points η, the map $\mathcal{N}^{\text{Pas}}_{x,\xi}$: $\mathcal{C} - S(\eta) \to \mathcal{C}$ defined as*

$$\mathcal{N}^{Pas}_{x,\eta}(\alpha,\omega,n) := \begin{cases} \big(\eta,\omega,n-[x,\omega,\alpha,\eta]\big) & \text{if} \quad \alpha \neq \eta \\ (\eta,\omega,n) & \text{if} \quad \alpha = \eta \end{cases}$$

makes the diagram

$$\begin{array}{ccc} \mathcal{G} - S(\eta) & \xrightarrow{\kappa_x} & \mathcal{C} - S(\eta) \\ \mathcal{N}^{\text{Pas}}_{\eta} \big\downarrow & & \big\downarrow \mathcal{N}^{\text{Pas}}_{x,\eta} \\ \mathcal{G} & \xrightarrow[\kappa_x]{} & \mathcal{C} \end{array}$$

commute. That is $\kappa_x \circ \mathcal{N}^{\text{Pas}}_{\eta} = \mathcal{N}^{\text{Pas}}_{x,\eta} \circ \kappa_x$ for all base points x and all boundary points η.

Proof. Since κ_x is invertible one has the identity $\mathcal{N}^{\text{Pas}}_{x,\eta} = \kappa_x \circ \mathcal{N}^{\text{Pas}}_{\eta} \circ \kappa_x^{-1} \overset{\text{Prop. 8.9}}{=} \kappa_x \circ \text{p} \circ \mathcal{N}^{\text{Fut}}_{\eta} \circ \text{p} \circ \kappa_x^{-1} = \widehat{\text{p}} \circ \mathcal{N}^{\text{Fut}}_{x,\eta} \circ \widehat{\text{p}}$, the last step follows with Proposition 8.6 and Theorem 7. Thus, the images of geodesics (α,ω,n) in x-coordinates under a new-past map to $\eta \in \mathcal{T}(\infty)$ (in x-coordinates) can be calculated.

$$\mathcal{N}^{\text{Pas}}_{x,\eta}(\alpha,\omega,n) \quad = \quad \widehat{\text{p}} \circ \mathcal{N}^{\text{Fut}}_{x,\eta} \circ \widehat{\text{p}}(\alpha,\omega,n) \quad = \quad \widehat{\text{p}} \circ \mathcal{N}^{\text{Fut}}_{x,\eta}(\omega,\alpha,-n).$$

If $\alpha \neq \eta$ this gives $\widehat{\text{p}}\big(\omega,\eta,-n+[x,\omega,\alpha,\eta]\big) = \big(\eta,\omega,n-[x,\omega,\alpha,\eta]\big)$. When $\alpha = \eta$ then we get $\mathcal{N}^{\text{Pas}}_{x,\eta}(\alpha,\omega,n) = (\alpha,\omega,n)$. □

8.3 The Unit Tangent Bundle

We want to construct a direct analog of the unit tangent bundle of a differentiable manifold. There, the unit tangent bundle is the union of unit tangent spaces at each point of the manifold. Every unit tangent space at a point q consists of the set of directions that a curve through q may assume, so it can be identified with S^d for the d-dimensional case.

In the setting of a tree, the unite tangent space at a vertex z would be the set of bi-infinite lines that run through z.

8.10 Definition (Unit Tangent Space)**.** The *unit tangent space* at a vertex $z \in \text{V}\mathcal{T}$ is defined as the set of lines

$$\text{T}^1_z\mathcal{T} := \big\{ \quad l \in \mathcal{V}_{\mathcal{T}} \quad : \quad z \in l \quad \big\}.$$

A vertex z lies on a bi-infinite line (η, ξ) if and only if it has distance zero to that line. So by Corollary 5.46 one can write

$$\mathrm{T}^1_z\mathcal{T} = \{ \quad (\eta,\xi) \in \mathcal{V}_{\mathcal{T}} \quad : \quad [z,\eta,z,\xi] = 0 \quad \}. \tag{8.5}$$

8.11 Definition (Unit Tangent Bundle)**.** The *unit tangent bundle* of a tree $\mathcal{T}$ is defined as

$$\mathrm{T}^1\mathcal{T} := \{ \quad (z,l) \quad : \quad l \in \mathrm{T}^1_z\mathcal{T}, \quad z \in \mathrm{V}(\mathcal{T}) \quad \}.$$

We define the *embedding onto the unit tangent bundle* as the map

$$\varsigma : \begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathrm{T}^1\mathcal{T} \\ g & \longmapsto & \big(g(0), \alpha(g), \omega(g)\big). \end{array} \tag{8.6}$$

For lines $(\eta, \xi) \in \mathcal{V}_{\mathcal{T}}$ and vertices $x \in \mathrm{V}(\mathcal{T})$ let

$$(\eta,\xi)_x$$

denote the unique geodesic with velocity (η, ξ) such that $(\eta,\xi)_x(0) = \mathfrak{b}(x,\eta,\xi)$.

Theorem 9. *The embedding onto the unit tangent bundle $\varsigma : \mathcal{G} \to \mathrm{T}^1\mathcal{T}$ is a bijection. The connection with x-coordinates for base points $x \in \mathrm{V}(\mathcal{T})$ to coordinates of the unit tangent bundle is given by*

$$\kappa_x \circ \varsigma^{-1}(z,\eta,\xi) = \big(\eta,\xi,[x,z,\eta,\xi]\big) \tag{8.7}$$

for all elements $(z,\eta,\xi) \in \mathrm{T}^1\mathcal{T}$ and by

$$\varsigma \circ \kappa_x^{-1}(\eta,\xi,n) = \Big(\big(\eta,\xi\big)_x(n), \eta, \xi\Big) \tag{8.8}$$

for all elements $(\eta,\xi,n) \in \mathcal{C}$.

Proof. At first note that $\varsigma(g) \in \mathrm{T}^1\mathcal{T}$, since $g(0) \in \big(\alpha(g), \omega(g)\big)$. When a base point $x \in \mathrm{V}(\mathcal{T})$ is chosen, the geodesic space is in bijection to the coordinate space via the map $\kappa_x : \mathcal{G} \to \mathcal{C}$, $\kappa_x(g) = \big(\alpha(g), \omega(g), [x, g(0), \alpha(g), \omega(g)]\big)$ for geodesics $g \in \mathcal{G}$ (Theorem 6).

We define $\gamma_x : \mathrm{T}^1\mathcal{T} \to \mathcal{C}$ by $\gamma_x(z,\eta,\xi) := \big(\eta,\xi,[x,z,\eta,\xi]\big)$. Since for all $g \in \mathcal{G}$ holds $\gamma_x \circ \varsigma(g) = \gamma_x\big(g(0),\alpha(g),\omega(g)\big) = \big(\alpha(g),\omega(g),[x,g(0),\alpha(g),\omega(g)]\big) = \kappa_x(g)$, one has the identity

$$\gamma_x \circ \varsigma = \kappa_x.$$

This shows also that γ_x has a right inverse, i.e. γ_x is surjective. A left inverse for γ_x is to be found.

Let $\phi_x : \mathcal{C} \to \mathrm{T}^1\mathcal{T}$ be defined as $\phi_x(\eta,\xi,n) := \big((\eta,\xi)_x(n),\eta,\xi\big)$. Then for all $(z,\eta,\xi) \in \mathrm{T}^1\mathcal{T}$ holds the equation $\phi_x \circ \gamma_x(z,\eta,\xi) = \phi_x\big(\eta,\xi,[x,z,\eta,\xi]\big) = \big((\eta,\xi)_x([x,z,\eta,\xi]),\eta,\xi\big)$, which implies

$$\phi_x \circ \gamma_x = \mathrm{Id}|_{\mathrm{T}^1\mathcal{T}},$$

if we can show that $(\eta,\xi)_x([x,z,\eta,\xi]) = z$ for all base points $x \in \mathrm{V}(\mathcal{T})$ and all elements $(z,\eta,\xi) \in \mathrm{T}^1\mathcal{T}$. One has $z = (\eta,\xi)_x(n)$ for some $n \in \mathbb{Z}$, so it remains to show that $\big[x,(\eta,\xi)_x(n),\eta,\xi\big] = n$.

$$\begin{aligned}\big[x,(\eta,\xi)_x(n),\eta,\xi\big] &\overset{(8.1)}{=} \big[x,(\eta,\xi)_x(0),\eta,\xi\big] + \big[(\eta,\xi)_x(0),(\eta,\xi)_x(n),\eta,\xi\big] \\ &\overset{\text{Lemma 5.42}}{=} \big[x,\mathfrak{b}(x,\eta,\xi),\eta,\xi\big] + n \overset{\text{Coro. 5.44}}{=} \big[x,x,\eta,\xi\big] + n \overset{(8.1)}{=} n.\end{aligned}$$

In conclusion, γ_x is invertible, hence ς is invertible and one obtains $\kappa_x \circ \varsigma^{-1} = \gamma_x$ as well as $\varsigma \circ \kappa_x^{-1} = \gamma_x^{-1} = \phi_x$. □

Chapter 9

Invariants of the Ideal Boundary

This chapter considers the behavior of invariants of quadrilaterals of boundary points of a tree that are available in every tree — thus called universal invariants. Together they comprise exactly the Klein type and the inner diameter of each quadrilateral. It turns out that there is only one independent invariant among them. This function is a complete invariant for the full automorphism group of a regular tree.

9.1 $U_C = \mathtt{diam} - 2 \cdot k_{(1\,3)(2\,4)} + (\sqrt{2} - 1) \cdot k_{(1\,4)(2\,3)}$

The vector space `UI` of universal invariants is introduced. It has dimension zero or three. Besides the base chosen for the definition, two more bases for this space are introduced that will find applications. In case of dimension three, the base consisting of *Delay, Concordance* and the *Inner Diameter* has (besides the well-sounding names) the advantage that the complete reducibility of the representation $S_4 \to \mathrm{GL}(\mathtt{UI})$ on the two subspaces spanned by $\{\mathtt{diam}\}$ and $\{\mathtt{del}, [\cdot]\}$ comes to daylight. The delay `del` has also a geometrical interpretation as presented in Section 9.3. The geometrical interpretation of the inner diameter `diam` is obvious. The third function in this collection, the concordance $[\cdot]$ will prove to have "nice" algebraic properties as indicated in the last chapter. A third base is used to identify a single invariant function $U_C \in \mathtt{UI}$ that contains the same information about a quadrilateral as the whole function space $\mathtt{UI}$ does.

For trees $\mathcal{T}$ we introduce the space of *quadrilaterals of boundary points*

$$\mathcal{T}_{\mathrm{Q}}^{\infty} := \mathcal{T}_{\mathrm{Q}} \cap \big(\mathcal{T}(\infty) \times \mathcal{T}(\infty) \times \mathcal{T}(\infty) \times \mathcal{T}(\infty)\big).$$

Clearly $\mathcal{T}_{\mathrm{Q}}^{\infty}$ is closed under S_4, i.e. $S_4(\mathcal{T}_{\mathrm{Q}}^{\infty}) = \{\sigma(A) : A \in \mathcal{T}_{\mathrm{Q}}^{\infty}, \sigma \in S_4\} = \mathcal{T}_{\mathrm{Q}}^{\infty}$. We introduce *distance functions* $\mathcal{T}_{\mathrm{Q}}^{\infty} \to \mathbb{N}_0$ defined for quadrilaterals $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$ as

$$\mathrm{U}_{i,j}(A) := \mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big)$$

for $i, j \in \{1,2,3,4\}$. These functions have been shown in Corollary 5.3 to be invariant under isometries: $\mathrm{U}_{i,j}(hA) = \mathrm{U}_{i,j}(A)$ for all quadrilaterals $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$, all isometries $h \in \mathrm{Is}(\mathcal{T})$ and for $i, j \in \{1,2,3,4\}$. There are redundancies (apart from the trivial ones $\mathrm{U}_{1,1} = \mathrm{U}_{2,2} = \mathrm{U}_{3,3} = \mathrm{U}_{4,4} = 0$ and $\mathrm{U}_{i,j} = \mathrm{U}_{j,i}$ for $i, j \in \{1,2,3,4\}$)

$$\mathrm{U}_{1,2} = \mathrm{U}_{3,4}, \quad \mathrm{U}_{1,3} = \mathrm{U}_{2,4}, \quad \text{and} \quad \mathrm{U}_{1,4} = \mathrm{U}_{2,3}. \tag{9.1}$$

by Corollary 5.25. The vector space generated by all distance functions

$$\mathtt{UI} := \Big\{ \sum_{i=2}^{4} \lambda_i \mathrm{U}_{1,i} : \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C} \Big\}$$

is called the *space of universal invariant functions on quadrilaterals of boundary points*[1].

Theorem 10. *If* $\mathcal{T}_{\mathrm{Q}}^{\infty}$ *has quadrilaterals of all three non-centered types then* $\dim(\mathtt{UI}) = 3$.

Proof. Say A is of type $(1\,2)(3\,4)$, B of type $(1\,3)(2\,4)$ and C of type $(1\,4)(2\,3)$. Assume that $\lambda_2 \cdot \mathrm{U}_{1,2} + \lambda_3 \cdot \mathrm{U}_{1,3} + \lambda_4 \cdot \mathrm{U}_{1,4} = 0$. This equation holds then necessarily for the quadrilaterals A, B, C as arguments. Thus

$$0 \;=\; \lambda_2 \cdot \mathrm{U}_{1,2}(A) + \lambda_3 \cdot \mathrm{U}_{1,3}(A) + \lambda_4 \cdot \mathrm{U}_{1,4}(A) = (\lambda_3 + \lambda_4) \cdot \mathtt{diam}(A).$$

Similarly $0 = (\lambda_2 + \lambda_4) \cdot \mathtt{diam}(B)$ and $0 = (\lambda_2 + \lambda_3) \cdot \mathtt{diam}(C)$. (See also Figure 9.1 for the expressions of $U_{i,j}$.) The diameters of A, B, C are positive,

[1] $\mathtt{UI}$ shall not be confused with Plaça Sant Iu in Barcelona, Catalonia.

so the three equations are equivalent to the equation

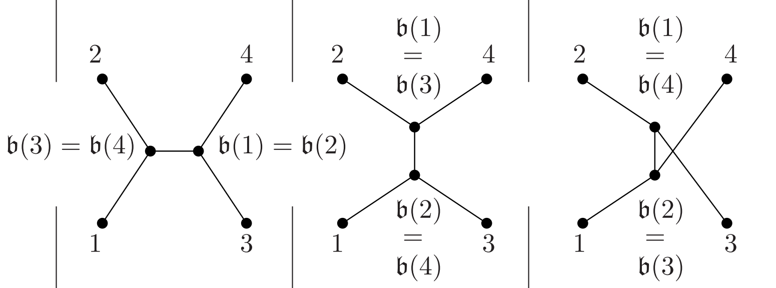
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

that implies $\lambda_2 = \lambda_3 = \lambda_4 = 0$ because

$$\det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = -1 \cdot \det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 2.$$

This proves that $\mathrm{U}_{1,2}, \mathrm{U}_{1,3}$ and $\mathrm{U}_{1,4}$ are linearly independent, hence form a base for `UI`. □

$A \in \mathcal{T}^\infty_{\mathrm{Q}}$ not centered		
type $(1\,2)(3\,4)$	type $(1\,3)(2\,4)$	type $(1\,4)(2\,3)$



	type $(1\,2)(3\,4)$	type $(1\,3)(2\,4)$	type $(1\,4)(2\,3)$
$\mathrm{U}_{1,2}(A)$	0	$\mathtt{diam}(A)$	$\mathtt{diam}(A)$
$\mathrm{U}_{1,3}(A)$	$\mathtt{diam}(A)$	0	$\mathtt{diam}(A)$
$\mathrm{U}_{1,4}(A)$	$\mathtt{diam}(A)$	$\mathtt{diam}(A)$	0

Figure 9.1: The distance functions $\mathrm{U}_{i,j}$ defined as $\mathrm{U}_{i,j}(A) = \mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big)$ for quadrilaterals A are written in terms of the inner diameter. Confer Proposition 5.22 for a proof of the expressions and Corollary 5.27 for the bifurcation configurations.

A few combinatorial results on the existence of quadrilaterals should be gathered now.

9.1 Definition (Geodesic complete trees)**.** A tree is called *geodesic complete* if each geodesic segment can be extended to a bi-infinite line.

9.2 Note. In view of a dynamical system consisting of $(\mathcal{G}, \mathrm{L})$, the emphasize on geodesic complete trees does not hide any difficulties. It rather distracts attention away from negligible vertices. A vertex that does not lie on a bi-infinite line has no relevance for the dynamics since it would never be visited.

9.3 Proposition. *Assume that $\mathcal{T}$ is a geodesic complete tree. Then $\mathcal{T}_{\mathrm{Q}}^{\infty}$ contains a quadrilateral of type $(1\,2)(3\,4)$ if and only if $\mathcal{T}$ has two vertices of degree greater than two. Otherwise all quadrilaterals of $\mathcal{T}_{\mathrm{Q}}^{\infty}$ are centered.*

Proof. We choose two distinct vertices $x \neq y$ of degree at least three and consider the segment $[x, y]$ of positive length. There are two vertices $z_1 \neq z_2$ adjacent to x but not included in $[x, y]$. There are two vertices $z_3 \neq z_4$ adjacent to y but not included in $[x, y]$. Since $\mathcal{T}$ is geodesic complete, four extensions

$$\begin{aligned} & (1, x][x, y] && \text{with} && z_1 \in (1, x], \\ & (2, x][x, y] && \text{with} && z_2 \in (2, x], \\ & [x, y][y, 3) && \text{with} && z_3 \in [y, 3), \\ \text{and} \quad & [x, y][y, 4) && \text{with} && z_4 \in [y, 4) \end{aligned}$$

can be chosen. It is clear that $A := (1, 2, 3, 4)$ is a quadrilateral: The ray $[x, 1)$ is uniquely defined by the vertex x and the boundary point 1. Hence $2 = 1$ implies $z_2 = z_1$. Similarly $3 \neq 4$. Since $[x, y]$ has positive length, $(1, 3)$, $(1, 4)$, $(2, 3)$ and $(2, 4)$ are lines. Therefore $A := (1, 2, 3, 4) \in \mathcal{T}_{\mathrm{Q}}^{\infty}$.

Moreover $[x, y] \subset (1, 3) \cap (2, 4)$ implies $|(1, 3) \cap (2, 4)| \geq 2$ so that Corollary 5.26 shows that A is not centered. Since $x \in (1, 2)$, $x \in (1, 3) \cap (2, 3) \cap (1, 4) \cap (2, 4)$ it follows $\mathfrak{b}(1, 2, 3) = x = \mathfrak{b}(1, 2, 4)$. Thus $\mathrm{Kl}(A) = (1\,2)(3\,4)$ by Corollary 5.27.

If $\mathcal{T}$ has at most one vertex of degree greater than two we may choose a quadrilateral $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$. For $k := \big[\mathfrak{b}(1)_A, 2_A\big)$, $l := \big[\mathfrak{b}(1)_A, 3_A\big)$ and $m := \big[\mathfrak{b}(1)_A, 3_A\big)$, the vertices $k(1), l(1)$ respectively $m(1)$ are included in the infinite lines k, l respectively m. These lines are mutually disjoint except for $k(0) = l(0) = m(0)$ by Lemma 4.9 `(c)`. This shows that $\mathfrak{b}(1)_A$ has degree at least three. Similarly the remaining bifurcations of A have degree at least three. Since $\mathcal{T}$ has at most one vertex of degree greater than two this shows that A is centered. □

9.4 Corollary. *If $\mathcal{T}$ is a tree then the following statements are equivalent.*

- *$\mathcal{T}_{\mathrm{Q}}^{\infty}$ has quadrilaterals of all three non-centered types.*
- *$\mathcal{T}_{\mathrm{Q}}^{\infty}$ has a non-centered quadrilateral.*
- $\dim(\mathtt{UI}) = 3$.

Otherwise $\dim(\mathtt{UI}) = 0$*. If $\mathcal{T}$ is geodesic complete then a fourth equivalent condition is that*

- *$\mathcal{T}$ has two vertices of degree greater than two.*

Proof. Since $\mathcal{T}_{\mathrm{Q}}^{\infty}$ is closed under S_4, the first two conditions are equivalent by Lemma 5.18. If they are wrong then all quadrilaterals of $\mathcal{T}_{\mathrm{Q}}^{\infty}$ are centered. Thus the distance functions $\mathrm{U}_{i,j}$ vanish identically on $\mathcal{T}_{\mathrm{Q}}^{\infty}$, so that $\mathtt{UI}$ has dimension zero. This proves indirectly that $\dim(\mathtt{UI}) = 3$ implies the existence of a non-centered quadrilateral in $\mathcal{T}_{\mathrm{Q}}^{\infty}$. Conversely, if there are non-centered quadrilaterals of all three types then $\dim(\mathtt{UI}) = 3$ by Theorem 10. The equivalence of the fourth statement is proved in Proposition 9.3. □

We check that S_4 acts on $\mathtt{UI}$ by the assignment $(\sigma, F) \mapsto \sigma * F$ where $\sigma * F$ is defined for quadrilaterals A as $\sigma * F(A) := F\big(\sigma^{-1}(A)\big)$.

9.5 Lemma. *For all $\sigma \in S_4$ and $i, j \in \{1,2,3,4\}$ holds $\sigma * \mathrm{U}_{i,j} = \mathrm{U}_{\sigma(i),\sigma(j)}$.*

Proof. For all quadrilaterals $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$ holds

$$\begin{aligned} \sigma * \mathrm{U}_{i,j}(A) &= \mathrm{U}_{i,j}(\sigma^{-1}A) = \mathrm{d}\big(\mathfrak{b}(i)_{\sigma^{-1}A}, \mathfrak{b}(j)_{\sigma^{-1}A}\big) \\ &\overset{\text{Prop. 5.5}}{=} \mathrm{d}\Big(\mathfrak{b}\big(\sigma(i)\big)_A, \mathfrak{b}\big(\sigma(j)\big)_A\Big) = \mathrm{U}_{\sigma(i),\sigma(j)}(A). \end{aligned}$$

□

The previous lemma shows that $\mathtt{UI}$ is closed under S_4, i.e. $\sigma * F \in \mathtt{UI}$ for all $\sigma \in S_4$ and $F \in \mathtt{UI}$. Hence this assignment defines a group action of S_4 on $\mathtt{UI}$. We define a map $\Phi : S_4 \to \mathrm{Perm}(\mathtt{UI})$ for $\sigma \in S_4$ by

$$\Phi(\sigma)(F) := \sigma * F \tag{9.2}$$

for all $F \in \mathtt{UI}$.

9.6 Proposition. *The map Φ is a group homomorphism $S_4 \to \mathrm{GL}(\mathtt{UI})$.*

Proof. We first check that for $\sigma \in S_4$ the map $\Phi(\sigma)$ defined by $\Phi(\sigma)F := \sigma * F$ for all $F \in \mathtt{UI}$ is linear. Let $F_1, F_2 \in \mathtt{UI}$, $\lambda_1, \lambda_2 \in \mathbb{C}$ and put $F := \lambda_1 F_1 + \lambda_2 F_2$. Then for all $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$ holds

$$\begin{aligned} \Phi(\sigma)(\lambda_1 F_1 + \lambda_2 F_2)(A) \quad &= \quad \Phi(\sigma)(F)(A) \quad = \quad \sigma * F(A) \\ = \quad F\big(\sigma^{-1}(A)\big) \quad &= \quad \lambda_1 F_1\big(\sigma^{-1}(A)\big) + \lambda_2 F_2\big(\sigma^{-1}(A)\big) \\ = \quad \lambda_1 \sigma * F_1(A) + \lambda_2 \sigma * F_1(A) \quad &= \quad \lambda_1 \Phi(\sigma)(F_1)(A) + \lambda_2 \Phi(\sigma)(F_2)(A). \end{aligned}$$

Secondly one has for all $\tau, \sigma \in S_4$ and $F \in \mathtt{UI}$

$$\begin{aligned} \Phi(\tau)\circ\Phi(\sigma)(F) \quad &= \quad \Phi(\tau)(\sigma * F) \quad = \quad \tau * (\sigma * F) \quad = \quad (\tau\sigma) * F \\ = \quad &\Phi(\tau\sigma)(F). \end{aligned}$$

In particular $\Phi(\sigma)\circ\Phi(\sigma^{-1})(F) = \Phi(\sigma\sigma^{-1})(F) = \Phi(\mathrm{Id})(F) = \mathrm{Id} * F = F$ for all $\sigma \in S_4$ and all $F \in \mathtt{UI}$. Thus $\Phi(\sigma) \in \mathrm{GL}(\mathtt{UI})$ for all $\sigma \in S_4$. The previous equation $\Phi(\tau)\circ\Phi(\sigma) = \Phi(\tau\sigma)$ also states that $\Phi : S_4 \to \mathrm{GL}(\mathtt{UI})$ is a group homomorphism. □

9.7 Note. Homomorphisms from a group to an operator group of a vector space are often called representations. We will later in this section make use of the representation Φ to approach questions about symmetries of universal invariant functions with tools from Linear Algebra.

We recall the definitions of two invariant functions and introduce a third one.

$$\begin{aligned} \mathtt{diam}(A) \quad &= \quad \max_{i,j\in\{1,2,3,4\}} \mathrm{d}\big(\mathfrak{b}(i)_A, \mathfrak{b}(j)_A\big), \\ [A] \quad &= \quad \vec{o}(A)\cdot\mathtt{diam}(A), \\ \mathtt{del}(A) \quad &:= \quad (2\,3) * \big[A\big] \quad \text{for} A \in \mathcal{T}_{\mathrm{Q}}^{\infty}. \end{aligned} \tag{9.3}$$

The inner diameter `diam` was introduced in Section 5.3, the concordance $[\cdot]$ was defined in Section 5.5. The function `del` is called *delay.*

9.8 Proposition. *If* $\dim(\mathtt{UI}) = 3$ *then the functions* $\mathtt{del}, [\cdot], \mathtt{diam}$ *form a base for* `UI`.

Proof. We verify the equations

$$\begin{aligned} \mathtt{diam} \quad &= \quad \tfrac{1}{2}(\mathrm{U}_{1,2} + \mathrm{U}_{1,3} + \mathrm{U}_{1,4}), \\ [\cdot] \quad &= \quad \mathrm{U}_{1,4} - \mathrm{U}_{1,3}, \\ \text{and} \quad \mathtt{del} \quad &= \quad \mathrm{U}_{1,4} - \mathrm{U}_{1,2}. \end{aligned} \tag{9.4}$$

The first equality is clear from Figure 9.1. The second one was shown in Proposition 5.45. The third equation is true because for all $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$ holds

$$\begin{aligned} \mathtt{del}(A) &= (2\,3) * [A] = \big[(2\,3)A\big] \\ &= \mathrm{U}_{1,4}\big((2\,3)(A)\big) - \mathrm{U}_{1,3}\big((2\,3)(A)\big) \overset{\text{Lemma 9.5}}{=} \mathrm{U}_{1,4}(A) - \mathrm{U}_{1,2}(A). \end{aligned}$$

The matrix T transforming coordinates with respect to the vectors $\mathtt{del}, [\cdot], \mathtt{diam}$ into coordinates with respect to the base $\mathrm{U}_{1,2}, \mathrm{U}_{1,3}, \mathrm{U}_{1,4}$ reads thus

$$T = \begin{pmatrix} -1 & 0 & \frac{1}{2} \\ 0 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \end{pmatrix}$$

and has determinant $\frac{3}{2}$. This shows that the three functions above indeed form a base for $\mathtt{UI}$. For completeness we state the inverse transformations

$$\begin{aligned} \mathrm{U}_{1,2} &= \tfrac{1}{3}(-2\mathtt{del} + [\cdot] + 2\mathtt{diam}), \\ \mathrm{U}_{1,3} &= \tfrac{1}{3}(\mathtt{del} - 2[\cdot] + 2\mathtt{diam}), \\ \text{and} \quad \mathrm{U}_{1,4} &= \tfrac{1}{3}(\mathtt{del} + [\cdot] + 2\mathtt{diam}). \end{aligned} \tag{9.5}$$

□

The advantage of the base $\mathtt{del}, [\cdot], \mathtt{diam}$ is that the representation $\Phi : S_4 \to \mathrm{GL}(\mathtt{UI})$ takes a simple form in matrix notation. Observe that S_4 is generated by the cycles $(1\,2)$ and $(1\,2\,3\,4)$.

9.9 Proposition. *Assume that* $\dim(\mathtt{UI}) = 3$. *Then the matrices of the linear operators* $\Phi(1\,2) \in \mathrm{GL}(\mathtt{UI})$ *respectively* $\Phi(1\,2\,3\,4) \in \mathrm{GL}(\mathtt{UI})$ *with respect to the base* $\mathtt{del}, [\cdot], \mathtt{diam}$ *of* $\mathtt{UI}$ *are given by*

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \quad \textit{respectively} \quad \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right).$$

Proof. For all $\sigma \in S_4$ holds $\sigma * \mathtt{diam} = \mathtt{diam}$ by Corollary 5.24, so the last column of both matrices has one at the third row and zero otherwise. One has $(1\,2) * [A] = \big[(1\,2)A\big] = -[A]$ for all $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$ by Proposition 5.38. Thus the second column of the matrix of $\Phi(1\,2)$ is $(0, -1, 0)^T$. Further

$$\begin{aligned} (1\,2) * \mathtt{del} &\overset{(9.4)}{=} (1\,2) * \mathrm{U}_{1,4} - (1\,2) * \mathrm{U}_{1,2} \overset{\text{Lemma 9.5}}{=} \mathrm{U}_{2,4} - \mathrm{U}_{2,1} \\ &\overset{(9.1)}{=} \mathrm{U}_{1,3} - \mathrm{U}_{1,2} = \mathrm{U}_{1,4} - \mathrm{U}_{1,2} - (\mathrm{U}_{1,4} - \mathrm{U}_{1,3}) \overset{(9.4)}{=} \mathtt{del} - [\cdot]. \end{aligned}$$

Thus the first column of $\Phi(1\,2)$ is $(1,-1,0)^T$.

$$\begin{aligned} (1\,2\,3\,4) * [\cdot] &\overset{(9.4)}{=} (1\,2\,3\,4) * \mathrm{U}_{1,4} - (1\,2\,3\,4) * \mathrm{U}_{1,3} \\ &\overset{\text{Lemma 9.5}}{=} \mathrm{U}_{2,1} - \mathrm{U}_{2,4} \overset{(9.1)}{=} \mathrm{U}_{1,2} - \mathrm{U}_{1,3} \\ &= \mathrm{U}_{1,4} - \mathrm{U}_{1,3} - (\mathrm{U}_{1,4} - \mathrm{U}_{1,2}) = [\cdot] - \mathtt{del}. \end{aligned}$$

Thus the second column of the matrix for $\Phi(1\,2\,3\,4)$ equals $(-1,1,0)^T$. From the last formula one obtains

$$[\cdot] \overset{\text{Prop. 5.38}}{=} (1\,3)(2\,4) * [\cdot] = (1\,2\,3\,4)^2 * [\cdot] = (1\,2\,3\,4) * [\cdot] - (1\,2\,3\,4) * \mathtt{del}.$$

Therefore $(1\,2\,3\,4) * \mathtt{del} = (1\,2\,3\,4) * [\cdot] - [\cdot] = -\mathtt{del}$ shows that the first column of the matrix for $\Phi(1\,2\,3\,4)$ is $(-1,0,0)^T$. □

9.10 Corollary. *The functions $F \in \mathtt{UI}$ that are invariant under all permutations $\sigma \in S_4$ consist exactly of the subspace generated by the inner diameter* $\mathtt{diam}$.

Proof. This is clear if $\dim(\mathtt{UI}) = 0$. Otherwise $\mathtt{del}, [\cdot], \mathtt{diam}$ form a base for $\mathtt{UI}$. Since $(1\,2)$ and $(1\,2\,3\,4)$ generate S_4, the condition that a function $F \in \mathtt{UI}$ is invariant under S_4 is equivalent to the conditions $(1\,2) * F = F$ and $(1\,2\,3\,4) * F = F$. By definition (9.2) this is equivalent to $\big(\Phi(1\,2) - \mathrm{Id}|_{\mathtt{UI}}\big)(F) = 0$ and $\big(\Phi(1\,2\,3\,4) - \mathrm{Id}|_{\mathtt{UI}}\big)(F) = 0$. For the introduced base, the operators of these equations have matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since the matrix

$$\begin{pmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \end{pmatrix}$$

has rank two, the solutions are given by the subspace $\mathbb{C} \cdot \mathtt{diam}$. □

9.11 Note. Given a group G and a complex vector space V, a representation $G \to \mathrm{GL}(V)$ is called *irreducible* if there is no proper non-zero subspace $W < V$, that is invariant under all group elements of G. In our case, the previous Corollary shows that $\langle \mathtt{diam} \rangle$ is the only invariant one-dimensional subspace of $\mathtt{UI}$. This shows that the representation $\Phi|_{\langle \mathtt{del}, [\cdot] \rangle} : S_4 \to \mathrm{GL}(\langle \mathtt{del}, [\cdot] \rangle)$ is irreducible. Moreover $\mathtt{UI}$ is the direct sum $\mathtt{UI} = \langle \mathtt{del}, [\cdot] \rangle \oplus \langle \mathtt{diam} \rangle$. A representation with these properties is called *completely reducible.* See [FC91] for concepts from representation theory.

If $\dim(\texttt{UI}) = 3$, another base for `UI` labeled by $V - \{\mathrm{Id}\}$ will prove useful. On one hand the base allows to link the action of S_4 on `UI` with the Klein type. This on the other hand will help to find an invariant function in `UI` that contains the same information on quadrilaterals as the whole space `UI`.

$$\begin{aligned} k_{(1\,2)(3\,4)} &:= \tfrac{1}{2}(-\mathrm{U}_{1,2} + \mathrm{U}_{1,3} + \mathrm{U}_{1,4}), \\ k_{(1\,3)(2\,4)} &:= \tfrac{1}{2}(\mathrm{U}_{1,2} - \mathrm{U}_{1,3} + \mathrm{U}_{1,4}), \\ \text{and}\quad k_{(1\,4)(2\,3)} &:= \tfrac{1}{2}(\mathrm{U}_{1,2} + \mathrm{U}_{1,3} - \mathrm{U}_{1,4}). \end{aligned} \tag{9.6}$$

for the elements $(1\,2)(3\,4), (1\,3)(2\,4)$ and $(1\,4)(2\,3)$ from the Klein 4-group V.

One can read out of Figure 9.1 the following descriptions for the functions k_a, $a \in V - \mathrm{Id}$. For all quadrilaterals $A \in \mathcal{T}^\infty_{\mathrm{Q}}$ holds

$$k_a(A) \quad = \quad \texttt{diam}(A) \cdot \delta\big(\mathrm{Kl}(A), a\big). \tag{9.7}$$

Here $\delta(a, b)$ denotes the Kronecker symbol for $a, b \in V$ that equals one if $a = b$ and zero otherwise.

9.12 Proposition. *If* $\dim(\texttt{UI}) = 3$ *then* $\big\{k_a\big\}_{a\in V-\{\mathrm{Id}\}}$ *is a base for* `UI`.

Proof. The matrix of coordinate change from the $k's$ ordered as in Equation (9.6) to $\mathrm{U}_{1,2}, \mathrm{U}_{1,3}, \mathrm{U}_{1,4}$ is given by

$$\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

and has determinant $\frac{1}{2}$. The inverse transformations are given by

$$\begin{aligned} \mathrm{U}_{1,2} &= k_{(1\,3)(2\,4)} + k_{(1\,4)(2\,3)}, \\ \mathrm{U}_{1,3} &= k_{(1\,4)(2\,3)} + k_{(1\,2)(3\,4)}, \\ \text{and}\quad \mathrm{U}_{1,4} &= k_{(1\,2)(3\,4)} + k_{(1\,3)(2\,4)}. \end{aligned} \tag{9.8}$$

□

9.13 Lemma. *For a in $V - \{\mathrm{Id}\}$ and $\sigma \in S_4$ holds $\sigma * k_a = k_{\alpha(\sigma)(a)}$. Here α is defined as in* Eqn. (5.1).

Proof. We choose any quadrilateral $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$. Then

$$\begin{aligned} \sigma * k_a(A) &= k_a(\sigma^{-1}A) = \texttt{diam}(\sigma^{-1}A) \cdot \delta\big(\mathrm{Kl}(\sigma^{-1}A), a\big) \\ &= \texttt{diam}(A) \cdot \delta\big(\sigma^{-1}\mathrm{Kl}(A)\sigma, a\big) = \texttt{diam}(A) \cdot \delta\big(\mathrm{Kl}(A), \sigma a \sigma^{-1}\big) \\ &= k_{\alpha(\sigma)(a)}(A). \end{aligned}$$

□

9.14 Proposition. *There is an isomorphism* $\gamma : \beta(S_4) \to \Phi(S_4)$ *such that* $\gamma \circ \beta = \Phi$. *Here* β *is defined as in* Prop. 5.19. *In particular* $V < \ker(\beta) = \ker(\Phi)$.

Proof. In view of Proposition A.4 it is sufficient to prove $\ker(\beta) = \ker(\Phi)$. We distinguish the two cases $\dim(\texttt{UI}) = 3$ and $\dim(\texttt{UI}) = 0$. Let $\dim(\texttt{UI}) = 3$. Then $\{k_a\}_{a \in V - \{\mathrm{Id}\}}$ is a base for UI (Prop. 9.12). Now for $\sigma \in S_4$ holds

$$\begin{aligned} \sigma \in \ker(\Phi) \quad &\Leftrightarrow \quad \Phi(\sigma)(k_a) = k_a \quad \text{for all} \quad a \in V - \{\mathrm{Id}\} \\ &\overset{(9.2)}{\Leftrightarrow} \quad \sigma * k_a = k_a \quad \text{for all} \quad a \in V - \{\mathrm{Id}\} \\ &\overset{\text{Lemma 9.13}}{\Leftrightarrow} \quad k_{\alpha(\sigma)(a)} = k_a \quad \text{for all} \quad a \in V - \{\mathrm{Id}\}. \end{aligned}$$

By Corollary 9.4 there are quadrilaterals of all three non-centered types, say $\mathrm{Kl}(A_a) = a$ for $a \in V - \mathrm{Id}$. So if $k_{\alpha(\sigma)(a)} = k_a$ for all $a \in V - \{\mathrm{Id}\}$ then

$$\begin{aligned} \texttt{diam}(A_a) \cdot \delta\big(a, \alpha(\sigma)(a)\big) &= k_{\alpha(\sigma)(a)}(A_a) = k_a(A_a) \\ &= \texttt{diam}(A_a) \cdot \delta\big(a, a\big) \end{aligned}$$

for all $a \in V - \{\mathrm{Id}\}$. Since $\texttt{diam}(A_a) > 0$ this implies that $\alpha(\sigma)(a) = a$ for all $a \in V - \{\mathrm{Id}\}$. Hence $\sigma \in \ker(\alpha) = V$. Conversely if $\sigma \in \ker(\alpha)$ then $k_{\alpha(\sigma)(a)} = k_a$ for all $a \in V - \{\mathrm{Id}\}$. So $\ker(\Phi) = V \overset{\text{Coro. 5.21}}{=} \ker(\beta)$.

Let $\dim(\texttt{UI}) = 0$. Then $\Phi(S_4) = \mathrm{Id}_{\mathrm{GL}(\texttt{UI})}$ and thus $\ker(\Phi) = S_4$. By Corollary 9.4 all quadrilaterals of $\mathcal{T}_{\mathrm{Q}}^{\infty}$ are centered thus $\ker(\Phi) = S_4 \overset{\text{Coro. 5.21}}{=} \ker(\beta)$. □

The set of all functions of UI produce a partition of $\mathcal{T}_{\mathrm{Q}}^{\infty}$ into classes of quadrilaterals which agree under all of these functions and that are preserved by isometries. The invariant functions $\{k_a\}_{a \in V - \{\mathrm{Id}\}}$ form a base for UI, hence two quadrilaterals A, B pertain to the same class if and only if they have the same Klein type and the same diameter. The classes can thus be sorted as displayed in Figure 9.2.

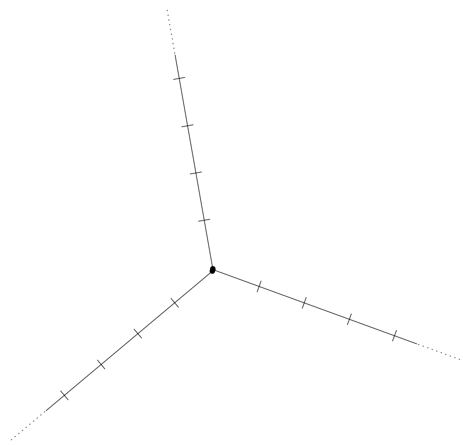


Figure 9.2: The classes of $\mathcal{T}_{\mathrm{Q}}^{\infty}$ consisting each one of quadrilaterals that have the same Klein type and the same inner diameter are assigned to a tree as follows. The tree consists of a vertex x of degree three with three rays r_a attached to x for $a \in V - \{\mathrm{Id}\}$. To the class of centered quadrilaterals (if not empty) the vertex x is assigned. To each other class (if not empty) with quadrilaterals of Klein type a and inner diameter n (for $a \in V - \{\mathrm{Id}\}, n \in \mathbb{N}$) is assigned a vertex y on the ray r_a, such that $\mathrm{d}(x, y) = n$.

There is a very informative function

$$U_C := \mathtt{diam} - 2 \cdot k_{(1\,3)(2\,4)} + (\sqrt{2} - 1) \cdot k_{(1\,4)(2\,3)} \tag{9.9}$$

in `UI` that carries all information about quadrilaterals that is contained in the whole space `UI`.

9.15 Proposition. *The value $U_C(A) \in \mathbb{C}$ for $U_C \in$ `UI` as defined in equation* (9.9) *determines the Klein type and the inner diameter of every quadrilateral $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$.*

Proof. If A is centered then $U_C(A) = 0$. If $\mathrm{Kl}(A) = (1\,2)(3\,4)$ then $U_C(A) = \mathtt{diam}(A) > 0$, if $\mathrm{Kl}(A) = (1\,3)(2\,4)$ then $U_C(A) = -\mathtt{diam}(A) < 0$ and if $\mathrm{Kl}(A) = (1\,4)(2\,3)$ then $U_C(A) = \sqrt{2} \cdot \mathtt{diam}(A) > 0$. The function U is constant on the classes of $\mathcal{T}_{\mathrm{Q}}^{\infty}$ that are defined to consist of quadrilaterals that have the same Klein type and the same inner diameter and induces hence a function $\widehat{U}_C$ from these classes to $\mathbb{C}$. We have to show that $\widehat{U}_C$ is injective, which is clearly the case if one can show that $\sqrt{2} \notin \mathbb{Q}$, the field of rational numbers. This is a classical proof to be found in the dawn of any course on number theory.

Assume that $\sqrt{2} = \frac{p}{q} \in \mathbb{Q}$ for a lowest terms fraction $\frac{p}{q}$ of integers $p, q \in \mathbb{Z}$. It follows that $p^2 = 2 \cdot q^2$, thus p^2 is divisible by 2. This implies that p is divisible by 2, so that q^2 is devisable by 2. Thus also q is devisable by 2 in contradiction to the assumption that $\frac{p}{q}$ is a lowest terms fraction. $\square$

9.16 Note. We are in a lucky position, we found a function in `UI` that determines the diameter and the Klein type. All invariant functions that are known so far (the orientation $\vec{o}$, the distances between bifurcations, hence the functions of `UI`) depend only on these two invariant quantities. We will see in the next section that there are no further invariants in the general setting. We can give examples — regular trees together with their full automorphism group — where no further invariants exist.

However, note that a more systematic search method for invariants should consider rings and algebras of functions (where products among the functions are allowed) rather than vector spaces only [Kra84].

Note also, that there is a simpler invariant function (not in `UI`) available that determines the Klein type and the inner diameter. Define $m_a(A) := \delta(\mathrm{Kl}(A), a)$ for $a \in V - \{\mathrm{Id}\}$. Then

$$U_D := 3 \cdot \mathtt{diam} - 2 \cdot m_{(1\,2)(3\,4)} - m_{(1\,3)(2\,4)}$$

is invariant. It is easily seen that $U_D : \mathcal{T}_{\mathrm{Q}}^{\infty} \to \mathbb{N}_0$ determines the values of Kl and `diam` through a case selection in the range $\mathbb{N}_0 \mod 3$.

9.2 Completeness of U_C for Regular Trees

Throughout this section we assume that $\mathcal{T}$ is a regular tree. We show that the invariant function U_C defined in Eqn. (9.9) is a *complete* invariant for the full automorphism group $\mathrm{Aut}(\mathcal{T})$ in the case that $\mathcal{T}$ is a regular tree. This means that for all quadrilaterals $A, B \in \mathcal{T}_{\mathrm{Q}}^{\infty}$ such that $U_C(B) = U_C(A)$ there is a morphism $h \in \mathrm{Aut}(\mathcal{T})$ such that $B = h(A)$. This condition is an equivalence relation and it is equivalent by Proposition 9.15 to the condition

$$A \sim B \quad :\Longleftrightarrow \quad \mathrm{Kl}(A) = \mathrm{Kl}(B) \quad \text{and} \quad \mathtt{diam}(A) = \mathtt{diam}(B). \tag{9.10}$$

Recall from A. Figà-Talamanca and C. Nebbia [FTN91] that the group $\mathrm{Aut}\,(\mathcal{T})$ acts transitively on $\mathrm{V}\mathcal{T}$. For each vertex $x \in \mathcal{T}$ the stabilizer $\mathrm{Aut}\,(\mathcal{T})_x$ of x acts transitively on the sets $F_n(x) := \{y \in \mathcal{T} : \mathrm{d}(x,y) = n\}$ for all $n \in \mathbb{N}_0$. Further $\mathrm{Aut}\,(\mathcal{T})_x$ acts transitively on the boundary $\mathcal{T}(\infty)$.

9.17 Lemma. *If $\mathcal{T}$ is a regular tree, $A, B \in \mathcal{T}_{\mathrm{Q}}^{\infty}$ and $A \sim B$ then there is an automorphism $h \in \mathrm{Aut}\,(\mathcal{T})$ such that $\mathfrak{b}(i)_B = \mathfrak{b}(i)_{h(A)}$ for $i = 1,2,3,4$.*

Proof. This is obvious if A and B are centered since $\mathrm{Aut}\,(\mathcal{T})$ acts transitively on the vertices of $\mathcal{T}$. In the case that $\mathrm{Kl}(A) = \mathrm{Kl}(B) = (1\,2)(3\,4)$ then one can define

$$\begin{aligned} x_1 &:= \mathfrak{b}(1)_A = \mathfrak{b}(2)_A, & x_2 &:= \mathfrak{b}(3)_A = \mathfrak{b}(4)_A, \\ y_1 &:= \mathfrak{b}(1)_B = \mathfrak{b}(2)_B \quad \text{and} & y_2 &:= \mathfrak{b}(3)_B = \mathfrak{b}(4)_B \end{aligned}$$

by Coro 5.27. Then by Prop. 5.22 holds $\mathrm{d}(x_1, x_2) = \mathtt{diam}(A) = \mathtt{diam}(B) = \mathrm{d}(y_1, y_2)$. Since $\mathrm{Aut}\,(\mathcal{T})$ acts transitively on the vertices of $\mathcal{T}$ there is an automorphism h_T such that $y_1 = h_T(x_1)$. As $\mathrm{d}\big(y_1, h_T(x_2)\big) = \mathrm{d}\big(h_T(x_1), h_T(x_2)\big) = \mathrm{d}(x_1, x_2) = \mathrm{d}(y_1, y_2)$, there is $h_R \in \mathrm{Aut}\,(\mathcal{T})_{y_1}$ with $y_2 = h_R h_T(x_2)$. Thus $y_1 = h_R(y_1) = h_R h_T(x_1)$ shows that $\mathfrak{b}(i)_B = h_R h_T\big(\mathfrak{b}(i)_A\big) = \mathfrak{b}(i)_{h_R h_T(A)}$ for $i = 1,2,3,4$.

In the general case that $\mathrm{Kl}(A) = \mathrm{Kl}(B) \neq \mathrm{Id}$ there is $\sigma \in S_4$ such that $\mathrm{Kl}(\sigma A) = \mathrm{Kl}(\sigma B) = (1\,2)(3\,4)$ (see the proof of Lemma 5.18). So there is $h \in \mathrm{Aut}\,(\mathcal{T})$ such that $\mathfrak{b}(i)_{\sigma(B)} = \mathfrak{b}(i)_{h \circ \sigma(A)}$ for $i = 1,2,3,4$. Thus

$$\begin{aligned} \mathfrak{b}(i)_B &= \mathfrak{b}(i)_{\sigma^{-1}\sigma B} \overset{\text{Prop. 5.5}}{=} \mathfrak{b}\big(\sigma(i)\big)_{\sigma B} = \mathfrak{b}\big(\sigma(i)\big)_{h \circ \sigma A} \\ &\overset{\text{Prop. 5.6}}{=} \mathfrak{b}\big(\sigma(i)\big)_{\sigma \circ h A} \overset{\text{Prop. 5.5}}{=} \mathfrak{b}(i)_{\sigma^{-1}\sigma h A} = \mathfrak{b}(i)_{hA} \end{aligned}$$

for $i = 1,2,3,4$. □

See [Wei04] for a definition and basic features of connected components of a graph.

9.18 Proposition. *If $e_1 := (x,y), e_2 := (x,z)$ are edges of a tree $\mathcal{T}$ such that $(x,z) = h(x,y)$ for some automorphism $h \in \mathrm{Aut}\,(\mathcal{T})$ then there is an automorphism $h^C \in \mathrm{Aut}\,(\mathcal{T})$ such that for all $p \in \mathcal{T}$ holds*

$$h^C(p) = \begin{cases} h(p) & \text{if} \quad p \in \mathcal{C}_{\mathcal{T}'}(y), \\ \mathrm{Id}(p) & \text{if} \quad p \in \mathcal{C}_{\mathcal{T}'}(x), \\ h^{-1}(p) & \text{else.} \end{cases}$$

For vertices $\widehat{x} \in \mathrm{V}(\mathcal{T})$, the graph $\mathcal{C}_{\mathcal{T}'}(\widehat{x})$ denotes the connected component of $\widehat{x}$ in the graph $\mathcal{T}'$ obtained from $\mathcal{T}$ by removing the edges $e_1, \overline{e_1}, e_2, \overline{e_2}$.

Proof. We first show that h^C is well defined, that there is no conflict in setting the definition. Since connected components are either equal or disjoint, it is more than sufficient to show that $\mathrm{V}\mathcal{T}$ is a disjoint union of $\mathrm{V}\mathcal{C}_{\mathcal{T}'}(x)$ and $\mathrm{V}\mathcal{C}_{\mathcal{T}'}(y) \cup \mathrm{V}\mathcal{C}_{\mathcal{T}'}(z)$. Observe that all connected components of $\mathcal{T}'$ are trees. If $x \in \mathcal{C}_{\mathcal{T}'}(y)$ then there is the geodesic from $[x, y]$ in the tree $\mathcal{C}_{\mathcal{T}'}(y)$. As $\mathcal{C}_{\mathcal{T}'}(y)$ is a subgraph of $\mathcal{T}$, $[x, y]$ is a geodesic in $\mathcal{T}$ thus has edge sequence (x, y), which is a contradiction to $(x, y) \notin \mathrm{E}\mathcal{T}'$. Similarly $x \notin \mathcal{C}_{\mathcal{T}'}(z)$, so the union is disjoint.

If p is any vertex of $\mathrm{V}\mathcal{T} = \mathrm{V}\mathcal{T}'$ then the second vertex $[x, p](1)$ of the geodesic $[x, p]$ equals y, z or another vertex. In the first two cases p is in $\mathcal{C}_{\mathcal{T}'}(y)$ or in $\mathcal{C}_{\mathcal{T}'}(z)$, in the latter it is in $\mathcal{C}_{\mathcal{T}'}(x)$. This shows that above union covers $\mathrm{V}\mathcal{T}$.

Why is h^C an automorphism? At first observe that $h^C\big(\mathcal{C}_{\mathcal{T}'}(y)\big) = \mathcal{C}_{\mathcal{T}'}(z)$. If $p \in \mathcal{C}_{\mathcal{T}'}(y)$, then $[x, p] = [x, y][y, p]$ so that the composition $[x, z][z, h^C(p)] = [h(x), h(y)][h(y), h(p)] = h\big([x, y][y, p]\big)$ is a geodesic and we get $h^C(p) \in \mathcal{C}_{\mathcal{T}'}(z)$. This shows $h^C\big(\mathcal{C}_{\mathcal{T}'}(y)\big) \subset \mathcal{C}_{\mathcal{T}'}(z)$. The morphism h_C as an automorphism of $\mathcal{T}$ is locally bijective. Then, since $\mathcal{C}_{\mathcal{T}'}(y)$ and $\mathcal{C}_{\mathcal{T}'}(z)$ are trees, h^C is an isomorphism from $\mathcal{C}_{\mathcal{T}'}(y)$ to $\mathcal{C}_{\mathcal{T}'}(z)$ (see [Wei04], Corollaries of local behavior).

Now there are two cases to distinguish, $\mathcal{C}_{\mathcal{T}'}(y) = \mathcal{C}_{\mathcal{T}'}(z)$ and $\mathcal{C}_{\mathcal{T}'}(y) \neq \mathcal{C}_{\mathcal{T}'}(z)$. In the latter case holds $h^C\big(\mathcal{C}_{\mathcal{T}'}(z)\big) = h^{-1}\big(\mathcal{C}_{\mathcal{T}'}(z)\big) = \mathcal{C}_{\mathcal{T}'}(y)$ because $\mathcal{T}'$ has three connected components. Thus $h^C|_{\mathcal{T}'}$ is an automorphism. If $\mathcal{C}_{\mathcal{T}'}(y) = \mathcal{C}_{\mathcal{T}'}(z)$ then $\mathcal{T}'$ has two components and h^C is an automorphism preserving both of them.

One has $h^C(x) = x$, $h^C(y) = h(y) = z$ and $h^C(z) = h^{-1}(z) = y$, so h^C maps all adjacent vertices to adjacent vertices and extends thus to $\mathcal{T}$. □

9.19 Definition (n-gons)**.** A *n-gon of boundary points* P is an ordered n-tuple of mutually distinct boundary points $P = (1_P, \ldots, n_P)$. A n-gon of boundary points is called *centered* if all its bifurcations are equal, that is $\mathfrak{b}(i_P, j_P, k_P) = \mathfrak{b}(l_P, m_P, n_P)$ whenever $\{i, j, k\}, \{l, m, n\} \subset \{1, \ldots, n\}$ are such that $|\{i, j, k\}| = |\{l, m, n\}| = 3$. The vertex that agrees with all bifurcations is then called the *center* of P.

For n-gons ($n \geq 3$) and isometries $h \in \mathrm{Is}\,(\mathcal{T})$ we set

$$h(P) = \big(1_{h(P)}, \dots, n_{h(P)}\big) := \big(h(1_P), \dots, h(n_P)\big).$$

Since isometries are injective, $h(P)$ is a n-gon.

9.20 Lemma. *If P is a centered n-gon ($n \geq 3$) with center x then $h(P)$ is centered with center $h(x)$ for all isometries $h \in \mathrm{Is}\,(\mathcal{T})$.*

Proof. Let $\mathfrak{b}(i_P, j_P, k_P)$ be any bifurcation for some subset $\{i, j, k\} \subset \{1, \dots, n\}$ and $|\{i, j, k\}| = 3$. Then $\mathfrak{b}\big(i_{h(P)}, j_{h(p)}, k_{h(P)}\big) = \mathfrak{b}\big(h(i_P), h(j_P), h(k_P)\big) \overset{\text{Prop. 4.8}}{=} h\big(\mathfrak{b}(i_P, j_P, k_P)\big) = h(x)$. ☐

9.21 Proposition. *If P, Q are two centered n-gons ($n \geq 3$) of boundary points in a regular tree $\mathcal{T}$ with the same center x then there is an automorphism $h \in \mathrm{Aut}\,(\mathcal{T})_x$ such that $Q = h(P)$.*

Proof. The affirmation follows by induction on $|F|$ from the following statement. Let N denote the set $\{1, \dots, n\}$.

> If R, S are two centered n-gons of boundary points in a regular tree that have the same centers and $i_S = i_R$ for all $i \in F$ for some $F \subset N$ with $|F| < n$ then there is an automorphism $h \in \mathrm{Aut}\,(\mathcal{T})$ such that $i_S = i_{h(R)}$ for all $i \in F' \supset F$ with $|F'| > |F|$ and such that the centers of R and $h(R)$ are equal.

For a proof of the statement, assume that for $F \subset N$ holds $i_S = i_R$ for all $i \in F$ and assume that for $l \in N - F$ holds $l_S \neq l_R$. We take x as the center of the two n-gons R and S. Further we define

$$y_i := [x, i_R)(1) \quad \text{and} \quad z_i := [x, i_S)(1)$$

for $i \in N$. By transitivity of $\mathrm{Aut}\,(\mathcal{T})_x$ on the boundary of $\mathcal{T}$, there is an automorphism $\widetilde{h} \in \mathrm{Aut}\,(\mathcal{T})_x$ such that $l_S = \widetilde{h}(l_R)$. Thus $z_l = [x, l_S)(1) = \big[\widetilde{h}(x), \widetilde{h}(l_R)\big)(1) = \widetilde{h}\big([x, 1_R)(1)\big) = \widetilde{h}(y_l)$. Now by Proposition 9.18 there is an isometry $h \in \mathrm{Aut}\,(\mathcal{T})_x$ such that

$$h|_{\mathcal{C}_{\mathcal{T}'}(x)} = \mathrm{Id}|_{\mathcal{C}_{\mathcal{T}'}(x)} \quad \text{and} \quad h|_{\mathcal{C}_{\mathcal{T}'}(y_l)} = \widetilde{h}|_{\mathcal{C}_{\mathcal{T}'}(y_l)}$$

for the connected components $\mathcal{C}_{\mathcal{T}'}(x)$ of x and $\mathcal{C}_{\mathcal{T}'}(y_l)$ of y_l in the graph $\mathcal{T}'$ that is obtained from $\mathcal{T}$ by removing the edges $(x, y_l), (y_l, x), (x, z_l)$ and (z_l, x).

Now $[x, l_R) = [x, y_l][y_l, l_R)$ so that $[y_l, l_R) \subset \mathcal{C}_{\mathcal{T}'}(y_l)$. This shows $[z_l, l_S) = \big[\widetilde{h}(y_l), \widetilde{h}(l_R)\big) = \widetilde{h}[y_l, l_R) = h[y_l, l_R) = \big[h(y_l), h(l_R)\big)$ and consequently $l_S = h(l_R) = l_{h(R)}$.

Observe that $y_1, \ldots, y_n$ are mutually distinct since $(i_R, j_R) = (i_R, x][x, j_R)$ for all $i \neq j$ in N. Similarly $z_1, \ldots, z_n$ are mutually distinct. For all $i \in F$ holds $z_i = [x, i_S)(1) = [x, i_R)(1) = y_i$, which implies

$$y_i \neq y_l \quad \text{and} \quad y_i \neq z_l$$

for all $i \in F$. This shows that $y_i \in \mathcal{C}_{\mathcal{T}'}(x)$ for all $i \in F$. The equality $[x, i_R) = [x, y_i][y_i, i_R)$ shows that $[x, i_R) \subset \mathcal{C}_{\mathcal{T}'}(x)$ for all $i \in F$. Thus $\big[h(x), h(i_R)\big) = h[x, i_R) = \mathrm{Id}[x, i_R) = [x, i_R)$ shows that $i_S = i_R = h(i_R) = i_{h(R)}$ for all $i \in F$. In conclusion, for the set $F' := F \cup \{l\}$ holds $i_S = h(i_R)$ for all $i \in F'$.

Finally, since $h \in \mathrm{Aut}\,(\mathcal{T})_x$, Lemma 9.20 shows that the center of $h(R)$ equals the center of R. □

Theorem 11. *If $\mathcal{T}$ is a regular tree and $A, B \in \mathcal{T}^\infty_{\mathrm{Q}}$ have the same bifurcations, i.e $\mathfrak{b}(i)_B = \mathfrak{b}(i)_A$ for $i = 1, 2, 3, 4$ then there is an automorphism $h \in \mathrm{Aut}\,(\mathcal{T})$ such that $B = h(A)$.*

Proof. If A and B are centered then the theorem is proved by Proposition 9.21 applied to the 4-gons A and B.

Assume that $\mathrm{Kl}(A) = \mathrm{Kl}(B) = (1\,2)(3\,4)$. Then $\mathfrak{b}(1)_A = \mathfrak{b}(1)_B = \mathfrak{b}(2)_A = \mathfrak{b}(2)_B \neq \mathfrak{b}(3)_A = \mathfrak{b}(3)_B = \mathfrak{b}(4)_A = \mathfrak{b}(4)_B$. We define $x = \mathfrak{b}(4)_A = \mathfrak{b}(4)_B$ as the center of the two triangles $(1_A, 2_A, 3_A)$ and $(1_B, 2_B, 3_B)$. Proposition 9.21 provides an isometry $h \in \mathrm{Aut}\,(\mathcal{T})_x$ such that $i_B = i_{h(A)}$ for $i = 1, 2, 3$. We check that $h(A)$ has the same bifurcations as A. By Prop. 5.2 follows $\mathfrak{b}(i)_{h(A)} = h\big(\mathfrak{b}(i)_A\big) = h(x) = x = \mathfrak{b}(i)_A$ for $i = 3, 4$. So it remains to check $\mathfrak{b}(1)_A = \mathfrak{b}(1)_{h(A)}$ and $\mathfrak{b}(2)_A = \mathfrak{b}(2)_{h(A)}$. Since $\mathfrak{b}(1)_{hA} = h\big(\mathfrak{b}(1)_A\big) \neq h\big(\mathfrak{b}(3)_A\big) = \mathfrak{b}(3)_{hA}$, Prop. 5.8 states that $\mathfrak{b}(1)_{hA} \in \big[\mathfrak{b}(3)_{hA}, 3_{hA}\big) = \big[\mathfrak{b}(3)_A, 3_B\big) = [x, 3_B)$. Also $\mathfrak{b}(1)_A = \mathfrak{b}(1)_B \in \big[\mathfrak{b}(3)_A, 3_B\big) = [x, 3_B)$ hence both $\mathfrak{b}(1)_{hA}$ and $\mathfrak{b}(1)_A$ lie on the same ray from x to 3_B. The distance equality $\mathrm{d}\big(x, \mathfrak{b}(1)_{hA}\big) = \mathrm{d}\Big(h(x), h\big(\mathfrak{b}(1)_A\big)\Big) = \mathrm{d}\Big(h\big(\mathfrak{b}(4)_A\big), h\big(\mathfrak{b}(1)_A\big)\Big) = \mathrm{d}\big(\mathfrak{b}(4)_A, \mathfrak{b}(1)_A\big) = \mathrm{d}\big(x, \mathfrak{b}(1)_A\big)$ shows thus $\mathfrak{b}(1)_{hA} = \mathfrak{b}(1)_A$. Also $\mathfrak{b}(2)_{hA} = h\big(\mathfrak{b}(2)_A\big) = h\big(\mathfrak{b}(1)_A\big) = \mathfrak{b}(1)_A = \mathfrak{b}(2)_A$.

By what has been shown in the previous paragraph, we assume that $\mathfrak{b}(i)_B = \mathfrak{b}(i)_A$ for $i = 1, 2, 3, 4$ and $i_B = i_A$ for $i = 1, 2, 3$. We are in the situation drawn in Figure 9.3.

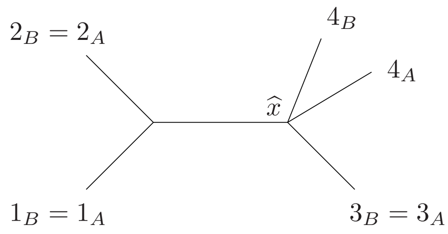


Figure 9.3: Identifying quadrilaterals $A \sim B$ of Klein type $(1\,2)(3\,4)$.

Define $\widehat{x} := \mathfrak{b}(1)_A = \mathfrak{b}(2)_A = \mathfrak{b}(1)_B = \mathfrak{b}(2)_B$ and set $y := [\widehat{x}, 4_A)(1)$ as well as $z := [\widehat{x}, 4_B)(1)$. By transitivity of $\operatorname{Aut}(\mathcal{T})_{\widehat{x}}$ on the boundary $\mathcal{T}(\infty)$ there is an automorphism $\widetilde{h} \in \operatorname{Aut}(\mathcal{T})_{\widehat{x}}$ such that $4_B = \widetilde{h}(4_A)$. This implies $z = [\widehat{x}, 4_B) = \big[\widetilde{h}(\widehat{x}), \widetilde{h}(4_A)\big)(1) = \widetilde{h}\big([\widehat{x}, 4_A)(1)\big) = \widetilde{h}(y)$. So by Prop 9.18 there is $h \in \operatorname{Aut}(\mathcal{T})_{\widehat{x}}\, x$ such that $h|_{\mathcal{C}_{\mathcal{T}'}(\widehat{x})} = \operatorname{Id}|_{\mathcal{C}_{\mathcal{T}'}(\widehat{x})}$ and $h|_{\mathcal{C}_{\mathcal{T}'}(y)} = \widetilde{h}|_{\mathcal{C}_{\mathcal{T}'}(y)}$ for the connected component $\mathcal{C}_{\mathcal{T}'}(\widehat{x})$ of $\widehat{x}$ and the connected component $\mathcal{C}_{\mathcal{T}'}(y)$ of y in the graph $\mathcal{T}'$ obtained from $\mathcal{T}$ by removing the edges $(\widehat{x}, y), (y, \widehat{x}), (\widehat{x}, z)$ and $(z, \widehat{x})$. By Lemma 4.9 on the partition of triangles, $[\widehat{x}, 1_A) \cap [\widehat{x}, 4_A) = \widehat{x}$ since $\widehat{x} = \mathfrak{b}(2)_A$. Also $[\widehat{x}, 1_A) = [\widehat{x}, 1_B)$ shows that $[\widehat{x}, 1_A) \cap [\widehat{x}, 4_B) = \widehat{x}$ because $\widehat{x} = \mathfrak{b}(2)_B$. Similarly $[\widehat{x}, 2_A) \cap [\widehat{x}, 4_A) = \widehat{x}$ since $\widehat{x} = \mathfrak{b}(1)_A$ and $[\widehat{x}, 2_A) \cap [\widehat{x}, 4_B) = \widehat{x}$ since $\widehat{x} = \mathfrak{b}(1)_B$. Finally $[\widehat{x}, 3_A) \cap [\widehat{x}, 4_A) = \widehat{x}$ and $[\widehat{x}, 3_A) \cap [\widehat{x}, 4_B) = \widehat{x}$. This shows that these rays are contained in $\mathcal{C}_{\mathcal{T}'}(\widehat{x})$ and as such are fixed by h. The final result is $i_B = i_{hA}$ for $i = 1, 2, 3, 4$, which is the same as to say $B = h(A)$.

For completeness consider two non-centered quadrilaterals $A \sim B$ in $\mathcal{T}_{\mathrm{Q}}^{\infty}$. By Lemma 5.18 there is $\sigma \in S_4$ such that $\sigma(A)$ and $\sigma(B)$ have Klein type $(1\,2)(3\,4)$. As shown above there is an automorphism $h \in \operatorname{Aut}(\mathcal{T})$ such that $\sigma(B) = h \circ \sigma(A)$. Prop. 5.6 shows that $h \circ \sigma(A) = \sigma \circ h(A)$ whence $\sigma(B) = \sigma \circ h(A)$ and therefore $B = h(A)$. □

9.22 Corollary. *If $\mathcal{T}$ is a regular tree, $A, B \in \mathcal{T}_{\mathrm{Q}}^{\infty}$ and $A \sim B$ then there is an automorphism $h \in \operatorname{Aut}(\mathcal{T})$ such that $B = h(A)$.*

Proof. This follows directly from Lemma 9.17 and Theorem 11. □

9.3 A Geometric Interpretation of Delay

The delay function `del` $: \mathcal{T}_{\mathrm{Q}}^{\infty} \mapsto \mathbb{Z}$ has a geometric interpretation. We choose a geodesic $g \in \mathcal{G}$ with velocity $\mathcal{V}(g) = (\alpha, \omega)$ and a second geodesic $h \in \mathcal{G}$ with velocity $\mathcal{V}(h) = (\eta, \xi)$ such that $\xi \neq \alpha$ and $\eta \neq \omega$.

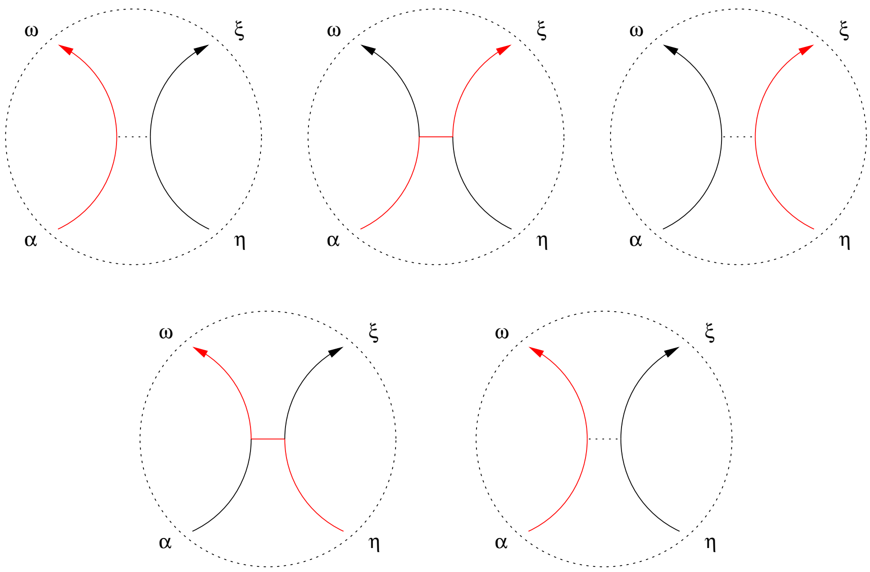


Figure 9.4: Geometric construction of the geodesic delay

Recall the concepts of strongly stable and strongly unstable manifolds from Section 8.2, as well as the definitions of the new-future maps $\mathcal{N}^{\mathrm{Fut}}$ and new past maps $\mathcal{N}^{\mathrm{Pas}}$. The image of g under the composed map

$$\mathcal{N}_{\alpha}^{\mathrm{Pas}} \circ \mathcal{N}_{\omega}^{\mathrm{Fut}} \circ \mathcal{N}_{\eta}^{\mathrm{Pas}} \circ \mathcal{N}_{\xi}^{\mathrm{Fut}} \tag{9.11}$$

is a geodesic with velocity (α, ω) like g. The geodesic g is transformed according to the diagrams (from left to right) in Figure 9.4. Each time a new future is assigned to the geodesic, the strongly unstable manifold is preserved and each time a new past is assigned to the geodesic, the strongly stable manifold is preserved.

9.23 Definition (Geodesic Delay)**.** If $g, h \in \mathcal{G}$ have velocities $\mathcal{V}(g) = (\alpha, \omega)$ and $\mathcal{V}(h) = (\eta, \xi)$ such that $\xi \neq \alpha$ and $\eta \neq \omega$ then a geodesic $g' := \mathcal{N}_{\alpha}^{\mathrm{Pas}} \circ$

$\mathcal{N}_\omega^{\rm Fut} \circ \mathcal{N}_\eta^{\rm Pas} \circ \mathcal{N}_\xi^{\rm Fut}(g)$ with velocity $\mathcal{V}(g') = (\alpha, \omega)$ is obtained. The unique integer

$$\texttt{Del}(g|h) := k \in \mathbb{Z} \quad \text{such that} \quad g' = \mathrm{L}^k(g)$$

is called the *geodesic delay* of g along h.

9.24 Note. The geodesic delay is well defined. The map defined in Eqn. (9.11) maps a geodesic g to a geodesic g' with the same velocity. Then Theorem 1 shows that $g' = L^k(g)$ for some $k \in \mathbb{Z}$.

Theorem 12. *If two geodesics $g, h \in \mathcal{G}$ obey the conditions $\omega(h) \neq \alpha(g)$ and $\alpha(h) \neq \omega(g)$ then*

$$\texttt{Del}(g|h) = \begin{cases} 0 & \text{if} \quad \alpha(h) = \alpha(g) \\ & \quad \text{or} \quad \omega(h) = \omega(g), \\ -2 \cdot \texttt{del}\big(\alpha(g), \omega(g), \alpha(h), \omega(h)\big) & \text{else.} \end{cases}$$

Proof. We set $(\alpha, \omega) := \mathcal{V}(g)$ and $(\eta, \xi) := \mathcal{V}(h)$ and choose a base point $x \in \mathrm{V}\mathcal{T}$. If $\eta \neq \alpha$ and $\xi \neq \omega$ then $(\alpha, \omega, \eta, \xi)$ is a quadrilateral and

$$\begin{aligned}
&\kappa_x \circ \mathcal{N}_\alpha^{\rm Pas} \circ \mathcal{N}_\omega^{\rm Fut} \circ \mathcal{N}_\eta^{\rm Pas} \circ \mathcal{N}_\xi^{\rm Fut}(g) \\
\overset{\text{Theorems 7 \& 8}}{=} \quad & \mathcal{N}_{x,\alpha}^{\rm Pas} \circ \mathcal{N}_{x,\omega}^{\rm Fut} \circ \mathcal{N}_{x,\eta}^{\rm Pas} \circ \mathcal{N}_{x,\xi}^{\rm Fut} \circ \kappa_x(g) \\
\overset{\text{Def. 8.1}}{=} \quad & \mathcal{N}_{x,\alpha}^{\rm Pas} \circ \mathcal{N}_{x,\omega}^{\rm Fut} \circ \mathcal{N}_{x,\eta}^{\rm Pas} \circ \mathcal{N}_{x,\xi}^{\rm Fut}\big(\alpha, \omega, \mathcal{X}_x(g)\big) \\
= \quad & \mathcal{N}_{x,\alpha}^{\rm Pas} \circ \mathcal{N}_{x,\omega}^{\rm Fut} \circ \mathcal{N}_{x,\eta}^{\rm Pas}\big(\alpha, \xi, \mathcal{X}_x(g) + [x, \alpha, \omega, \xi]\big) \\
= \quad & \mathcal{N}_{x,\alpha}^{\rm Pas} \circ \mathcal{N}_{x,\omega}^{\rm Fut}\big(\eta, \xi, \mathcal{X}_x(g) + [x, \alpha, \omega, \xi] - [x, \xi, \alpha, \eta]\big) \\
= \quad & \mathcal{N}_{x,\alpha}^{\rm Pas}\big(\eta, \omega, \mathcal{X}_x(g) + [x, \alpha, \omega, \xi] - [x, \xi, \alpha, \eta] + [x, \eta, \xi, \omega]\big) \\
= \quad & \big(\alpha, \omega, \mathcal{X}_x(g) + [x, \alpha, \omega, \xi] - [x, \xi, \alpha, \eta] + [x, \eta, \xi, \omega] - [x, \omega, \eta, \alpha]\big) \\
\overset{(8.1)}{=} \quad & \big(\alpha, \omega, \mathcal{X}_x(g) - [\alpha, x, \omega, \xi] - [x, \eta, \omega, \xi] - [\alpha, \eta, \omega, x] - [\alpha, \eta, x, \xi]\big) \\
\overset{(8.1)}{=} \quad & \big(\alpha, \omega, \mathcal{X}_x(g) - [\alpha, \eta, \omega, \xi] - [\alpha, \eta, \omega, \xi]\big) \\
\overset{(9.3)}{=} \quad & \big(\alpha, \omega, \mathcal{X}_x(g) - 2 \cdot \texttt{del}(\alpha, \omega, \eta, \xi)\big) \\
\overset{(8.2)}{=} \quad & \widehat{\mathrm{L}}^{-2 \cdot \texttt{del}(\alpha, \omega, \eta, \xi)}\big(\alpha, \omega, \mathcal{X}_x(g)\big) \\
\overset{\text{Def. 8.1}}{=} \quad & \widehat{\mathrm{L}}^{-2 \cdot \texttt{del}(\alpha, \omega, \eta, \xi)} \circ \kappa_x(g) \\
\overset{\text{Theorem 6}}{=} \quad & \kappa_x \circ \mathrm{L}^{-2 \cdot \texttt{del}(\alpha, \omega, \eta, \xi)}(g).
\end{aligned}$$

This implies $\mathcal{N}_\alpha^{\rm Pas} \circ \mathcal{N}_\omega^{\rm Fut} \circ \mathcal{N}_\eta^{\rm Pas} \circ \mathcal{N}_\xi^{\rm Fut}(g) = \mathrm{L}^{-2 \cdot \texttt{del}(\alpha, \omega, \eta, \xi)}(g)$ because κ_x is bijective by Theorem 6. Thus $\texttt{Del}(g|h) = -2 \cdot \texttt{del}(\alpha, \omega, \eta, \xi)$. If $\eta = \alpha$ then shorter as before one obtains

$$\begin{aligned}
&\kappa_x \circ \mathcal{N}^{\mathrm{Pas}}_{\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{\omega} \circ \mathcal{N}^{\mathrm{Pas}}_{\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{\xi}(g) \\
&\quad = \quad \mathcal{N}^{\mathrm{Pas}}_{x,\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{x,\omega} \circ \mathcal{N}^{\mathrm{Pas}}_{x,\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{x,\xi} \circ \kappa_x(g) \\
&\quad = \quad \mathcal{N}^{\mathrm{Pas}}_{x,\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{x,\omega} \circ \mathcal{N}^{\mathrm{Pas}}_{x,\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{x,\xi}\big(\alpha, \omega, \mathcal{X}_x(g)\big) \\
&\quad = \quad \mathcal{N}^{\mathrm{Pas}}_{x,\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{x,\omega} \circ \mathcal{N}^{\mathrm{Pas}}_{x,\alpha}\big(\alpha, \xi, \mathcal{X}_x(g) + [x, \alpha, \omega, \xi]\big) \\
&\quad = \quad \mathcal{N}^{\mathrm{Pas}}_{x,\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{x,\omega}\big(\alpha, \xi, \mathcal{X}_x(g) + [x, \alpha, \omega, \xi]\big) \\
&\quad = \quad \mathcal{N}^{\mathrm{Pas}}_{x,\alpha}\big(\alpha, \omega, \mathcal{X}_x(g) + [x, \alpha, \omega, \xi] + [x, \alpha, \xi, \omega]\big) \\
&\quad = \quad \big(\alpha, \omega, \mathcal{X}_x(g) + [x, \alpha, \omega, \xi] + [x, \alpha, \xi, \omega]\big) \\
&\quad \overset{(8.1)}{=} \quad \big(\alpha, \omega, \mathcal{X}_x(g)\big) \\
&\quad = \quad \kappa_x(g),
\end{aligned}$$

and this shows that $\mathcal{N}^{\mathrm{Pas}}_{\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{\omega} \circ \mathcal{N}^{\mathrm{Pas}}_{\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{\xi}(g) = g = L^0(g)$ whence $\mathtt{Del}(g|h) = 0$. For the case that $\xi = \omega$, note that $\mathcal{N}^{\mathrm{Fut}}_{x,\omega}(g') = g'$ and $\mathcal{N}^{\mathrm{Fut}}_{x,\omega}(g) = g$. Thus

$$\begin{aligned}
&\mathcal{N}^{\mathrm{Pas}}_{\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{\omega} \circ \mathcal{N}^{\mathrm{Pas}}_{\eta} \circ \mathcal{N}^{\mathrm{Fut}}_{\omega}(g) \\
&\quad = \quad \mathcal{N}^{\mathrm{Fut}}_{\omega} \circ \mathcal{N}^{\mathrm{Pas}}_{\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{\omega} \circ \mathcal{N}^{\mathrm{Pas}}_{\eta}(g) \\
&\quad \overset{\text{Lemma 8.5}}{=} \quad \mathcal{N}^{\mathrm{Fut}}_{\omega} \circ \mathcal{N}^{\mathrm{Pas}}_{\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{\omega} \circ \mathcal{N}^{\mathrm{Pas}}_{\eta} \circ \mathrm{p} \circ \mathrm{p}(g) \\
&\quad \overset{\text{Prop. 8.9}}{=} \quad \mathrm{p} \circ \mathcal{N}^{\mathrm{Pas}}_{\omega} \circ \mathcal{N}^{\mathrm{Fut}}_{\alpha} \circ \mathcal{N}^{\mathrm{Pas}}_{\omega} \circ \mathcal{N}^{\mathrm{Fut}}_{\eta} \circ \mathrm{p}(g).
\end{aligned}$$

Since $\mathrm{p}(g)$ has velocity (ω, α) by Prop. 8.6, this equals by the previous case $\mathrm{p} \circ \mathrm{p}(g) = g$. Thus $\mathtt{Del}(g|h) = 0$. □

It seems somehow arbitrary for the construction of the geodesic delay to first assign a new future, then a new past and so on. We consider a geodesic g with velocity $\mathcal{V}(g) = (\alpha, \omega)$ and a geodesic h with velocity (η, ξ). The image g' of g under the composed map

$$\mathcal{N}^{\mathrm{Fut}}_{\omega} \circ \mathcal{N}^{\mathrm{Pas}}_{\alpha} \circ \mathcal{N}^{\mathrm{Fut}}_{\xi} \circ \mathcal{N}^{\mathrm{Pas}}_{\eta} \tag{9.12}$$

is a geodesic with velocity (α, ω). Similarly to the case of the geodesic delay this identifies a unique integer $\overline{\mathtt{Del}}(g|h) \in \mathbb{Z}$ such that

$$g' = \mathrm{L}^{\overline{\mathtt{Del}}(g|h)}(g).$$

9.25 Corollary. *If two geodesics $g, h \in \mathcal{G}$ obey the conditions $\omega(h) \neq \alpha(g)$ and $\alpha(h) \neq \omega(g)$ then $\overline{\mathtt{Del}}(g|h) = -\mathtt{Del}(g|h)$.*

Proof. We put $\alpha := \alpha(g), \omega := \omega(g), \eta := \alpha(h)$ and $\xi := \omega(h)$. Then

$$\begin{aligned}
&\mathcal{N}_{\omega}^{\mathrm{Fut}} \circ \mathcal{N}_{\alpha}^{\mathrm{Pas}} \circ \mathcal{N}_{\xi}^{\mathrm{Fut}} \circ \mathcal{N}_{\eta}^{\mathrm{Pas}}(g) \\
\overset{\text{Lemma 8.5}}{=} \quad & \mathcal{N}_{\omega}^{\mathrm{Fut}} \circ \mathcal{N}_{\alpha}^{\mathrm{Pas}} \circ \mathcal{N}_{\xi}^{\mathrm{Fut}} \circ \mathcal{N}_{\eta}^{\mathrm{Pas}} \circ \mathrm{p} \circ \mathrm{p}(g) \\
\overset{\text{Prop. 8.9}}{=} \quad & \mathrm{p} \circ \mathcal{N}_{\omega}^{\mathrm{Pas}} \circ \mathcal{N}_{\alpha}^{\mathrm{Fut}} \circ \mathcal{N}_{\xi}^{\mathrm{Pas}} \circ \mathcal{N}_{\eta}^{\mathrm{Fut}} \circ \mathrm{p}(g) \\
\overset{\text{Theorem 12}}{=} \quad & \mathrm{p} \circ \mathrm{L}^{\mathtt{Del}(p(g)|p(h))} \circ \mathrm{p}(g) \\
= \quad & \mathrm{p} \circ \mathrm{p} \circ \mathrm{L}^{-\mathtt{Del}(p(g)|p(h))}(g) \\
= \quad & \mathrm{L}^{-\mathtt{Del}(p(g)|p(h))}(g).
\end{aligned}$$

Compared to Theorem 12 this shows that $\overline{\mathtt{Del}}(g|h) = -\mathtt{Del}(p(g)|p(h)) = 2 \cdot \mathtt{del}(\omega, \alpha, \xi, \eta) \overset{\text{Prop. 9.14}}{=} 2 \cdot \mathtt{del}(\alpha, \omega, \eta, \xi) = -\mathtt{Del}(g|h)$ if $(\alpha, \omega, \eta, \xi)$ is a quadrilateral. Otherwise it follows that $\eta = \alpha$ or $\xi = \omega$. Then $\overline{\mathtt{Del}}(g|h) = -\mathtt{Del}(p(g)|p(h)) = 0$ in accordance with $\mathtt{Del}(g|h) = 0$. □

Not only has the generation of the delay a geometric interpretation. A geometric interpretation for the values of the delay, that emphasizes the two involved lines, is also available. Let two lines $\langle 1,2\rangle, \langle 3,4\rangle$ have an empty intersection. Their *distance* is defined as

$$\mathrm{d}\big(\langle 1,2\rangle, \langle 3,4\rangle\big) := \min_{\substack{x\in\langle 1,2\rangle \\ y\in\langle 3,4\rangle}} \mathrm{d}(x,y).$$

9.26 Proposition. *If two lines $\langle 1,2\rangle, \langle 3,4\rangle$ have empty intersection, then a unique pair of vertices $a \in \langle 1,2\rangle$ and $b \in \langle 3,4\rangle$ exists such that $\mathrm{d}(a,b) = \mathrm{d}\big(\langle 1,2\rangle, \langle 3,4\rangle\big)$. Moreover $a = \mathfrak{b}(1,2,4) = \mathfrak{b}(1,2,3)$ and $b = \mathfrak{b}(2,3,4) = \mathfrak{b}(1,3,4)$.*

Proof. A pair $a \in \langle 1,2\rangle$, $b \in \langle 3,4\rangle$ in minimal distance exists by well-ordering of the natural numbers. If one such pair a, b is chosen, then a is a vertex on $\langle 1,2\rangle$ that has minimal distance to the vertex b. So by Lemma 4.10 holds $a = \mathfrak{b}(b,1,2)$. Similarly $b = \mathfrak{b}(a,3,4)$. Thus all compositions $\langle 1,a][a,b]$, $\langle 2,a][a,b]$, $[a,b][b,3\rangle$, and $[a,b][b,4\rangle$ are lines. Since $a \neq b$ it follows that $a,b \in \langle 1,3\rangle \cap \langle 1,4\rangle \cap \langle 2,3\rangle \cap \langle 2,4\rangle$. As $a \in \langle 1,2\rangle$, the Bifurcation Lemma shows that $a = \mathfrak{b}(1,2,3) = \mathfrak{b}(1,2,4)$. Finally $b \in \langle 3,4\rangle$ implies $b = \mathfrak{b}(1,3,4) = \mathfrak{b}(2,3,4)$. □

9.27 Proposition. Figure 9.5 *displays the correct values of the delay function* `del`.

Proof. From Eqn. (9.4) we know that $\mathtt{del} = \mathrm{U}_{1,4} - \mathrm{U}_{1,2}$. Then by comparison to Figure 9.1 the expressions for the delay in terms of the inner diameter $\mathtt{diam}$ becomes apparent. If $A = (\alpha, \omega, \eta, \xi)$ has Klein type (1 2)(3 4) then by Corollary 5.27 hold $(\alpha, \omega) \cap (\eta, \xi) = \emptyset$ and $\mathfrak{b}(\omega, \eta, \xi) \neq \mathfrak{b}(\alpha, \omega, \xi)$. Thus

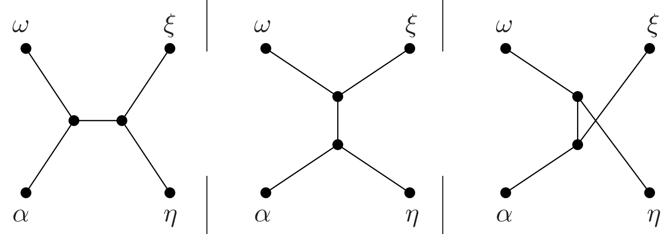
$$\mathtt{diam}(A) \overset{\text{Prop. 5.22}}{=} \mathrm{d}\big(\mathfrak{b}(\omega, \eta, \xi), \mathfrak{b}(\alpha, \omega, \xi)\big) \overset{\text{Prop. 9.26}}{=} \mathrm{d}\big((\alpha, \omega), (\eta, \xi)\big).$$

If $A = (\alpha, \omega, \eta, \xi)$ has Klein type (1 4)(2 3) then one has $(\alpha, \omega) \cap (\eta, \xi) \neq \emptyset$ and $\mathfrak{b}(\alpha, \eta, \xi) \neq \mathfrak{b}(\omega, \eta, \xi)$. Thus

$$\begin{aligned} \mathtt{diam}(A) &= \mathrm{d}\big(\mathfrak{b}(\alpha, \eta, \xi), \mathfrak{b}(\omega, \eta, \xi)\big) = \mathrm{len}\Big(\big[\mathfrak{b}(\alpha, \eta, \xi), \mathfrak{b}(\omega, \eta, \xi)\big]\Big) \\ &\overset{\text{Theorem 4}}{=} \mathrm{len}\big((\alpha, \omega) \cap (\eta, \xi)\big) \end{aligned}$$

in a slight abuse of notation, since the intersection of two lines is defined as a vertex set. $\square$

$A = (\alpha, \omega, \eta, \xi) \in \mathcal{T}_{\mathrm{Q}}^{\infty}$ not centered		
type (1 2)(3 4)	type (1 3)(2 4)	type (1 4)(2 3)



	type (1 2)(3 4)	type (1 3)(2 4)	type (1 4)(2 3)
$\mathtt{del}(A) =$	$\mathtt{diam}(A)$	0	$-\mathtt{diam}(A)$
$\mathtt{del}(A) =$	$\mathrm{d}\big((\alpha, \omega), (\eta, \xi)\big)$	0	$-\mathrm{len}\big((\alpha, \omega) \cap (\eta, \xi)\big)$

Figure 9.5: Geometric interpretation of the delay

Chapter 10

Prospects

There are many directions for further development. One could try to find more invariants — or one could try to find out more about invariants.

The first direction will depend on an explicit restriction to clear described trees and groups acting upon, since in the general setting there is only one invariant as shown in Section 9.2. We will give an example in the next section.

The second direction could be inspired by transcriptions from the analysis of geodesic flows on compact, connected, negatively curved Riemannian manifolds as motivated in the introduction, Chapter 1. The last two sections of the present chapter offer expectations that exhibit a kind of duality. The concordance may be interpreted as a *cross ratio*. On the other hand one can hope to see the same function as a *symmetric 2-form* on a (to be defined) tangent bundle $\mathrm{T}\mathcal{T}$ as an extension of the unit tangent bundle $\mathrm{T}^1\mathcal{T}$.

10.1 Trees with more Invariants

The class of semi-regular trees offers an easy example for an additional invariant.

A *p, q-semi-regular tree* is a tree that has only vertices of degrees $p, q \in \mathbb{N}$ such that for adjacent pairs of vertices x and y one of the vertices has degree p, the other has degree q.

If $\mathcal{T}$ is a p, q-semi-regular tree with $p \neq q$ and $p, q > 2$ then there are additional invariant functions ($i = 1, 2, 3, 4$)

$$\deg^i(A) := \begin{cases} p & \text{if} \quad \deg\big(\mathfrak{b}(i)_A\big) = p \\ q & \text{if} \quad \deg\big(\mathfrak{b}(i)_A\big) = q \end{cases} \tag{10.1}$$

defined for $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$. The functions are invariant under automorphisms. By Proposition 5.2, holds $\mathfrak{b}(i)_{h(A)} = h\big(\mathfrak{b}(i)_A\big)$ for $i = 1, 2, 3, 4$, all quadrilaterals $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$ and all $h \in \mathrm{Aut}\,(\mathcal{T})$. Since every automorphism preserves the classes of vertices with the same degree, one has $\deg\big(\mathfrak{b}(i)_{h(A)}\big) = \deg\Big(h\big(\mathfrak{b}(i)_A\big)\Big) = \deg\big(\mathfrak{b}(i)_A\big)$.

On the other hand, for each vertex $x \in \mathrm{V}(\mathcal{T})$ there is a quadrilateral A that has the bifurcation $\mathfrak{b}(1)_A$ located at x. One needed to show that $\mathcal{T}$ is geodesic complete (which should not be difficult). The proof of Proposition 9.3 contains then a proof of existence of such a quadrilateral. The degree of a bifurcation is new information that can not be expressed by the diameter or the Klein type of A.

In the same spirit one could investigate a tree with a group of isometries that does not act transitively on the vertices. In view of Proposition 9.21 a second approach would be to consider groups that have vertex stabilizers which do not act transitively on the boundary of the tree. It would be interesting to know how transitivity for vertices and transitivity for boundary points are related.

Considering a $n, 2$-semi-regular tree $\mathcal{T}$ for $n > 2$, every isometry still preserves the two classes of vertices with the same degree. But the invariants from Eqn. (10.1) are here all constant: since the bifurcation of a triangle of boundary points has at least degree three (see the proof of Prop. 9.3), it follows $\deg^i(A) = 3$ for all $A \in \mathcal{T}_{\mathrm{Q}}^{\infty}$ for all $i = 1, 2, 3, 4$. So the functions $\deg^1, \deg^2, \deg^3$ and $\deg^4$ do not contribute with a new invariant in the case of a $n, 2$-semi-regular tree.

10.2 Concordance as a Cross Ratio?

In the article [Ham99], U. Hamenstädt considers compact, connected, negatively curved Riemannian manifolds (M, g) with a universal covering $(\widetilde{M}, \widetilde{g})$ and its ideal boundary $\partial\widetilde{M}$. She relates a certain cross ratio, the cross ratio of the length cocycle of the metric $\widetilde{g}$, to the symplectic structure on the space of geodesics $\mathcal{G}\widetilde{M}$.

This motivates a search for cross ratios on trees. First let us state the definition from [Ham99].

10.1 Definition. A *generalized cross ratio* is a Hölder continuous positive function Cr on the space of quadruples of pairwise distinct points in $\partial\widetilde{M}$ with the following properties:

1. Cr is invariant under the action of the fundamental group on $\left(\partial\widetilde{M}\right)^4$;
2. $Cr(\xi,\xi',\eta,\eta') = Cr(\xi',\xi,\eta,\eta')^{-1}$;
3. $Cr(\xi,\xi',\eta,\eta') = Cr(\eta,\eta',\xi,\xi')$;
4. $Cr(\xi,\xi',\eta,\eta')Cr(\xi',\xi'',\eta,\eta') = Cr(\xi,\xi'',\eta,\eta')$;
5. $Cr(\xi,\xi',\eta,\eta')Cr(\xi',\eta,\xi,\eta')Cr(\eta,\xi,\xi'\eta') = 1$.

A comparison to the following relations for the concordance $[\cdot] : \mathcal{T}_{\mathrm{Q}}^{\infty} \to \mathbb{Z}$

1. $[\cdot]$ is invariant under the action of groups of isometries on $\mathcal{T}_{\mathrm{Q}}^{\infty}$; Corollary 5.31
2. $[\xi,\xi',\eta,\eta'] = -[\xi',\xi,\eta,\eta']$; Proposition 5.32
3. $[\xi,\xi',\eta,\eta'] = [\eta,\eta',\xi,\xi']$; Proposition 5.32
4. $[\xi,\xi',\eta,\eta'] + [\xi',\xi'',\eta,\eta'] = [\xi,\xi'',\eta,\eta']$; Theorem 5
5. $[\xi,\xi',\eta,\eta'] + [\xi',\eta,\xi,\eta'] + [\eta,\xi,\xi'\eta'] = 0.$ Proposition 5.32

shows that any positive real number taken to the power of concordance, say the function

$$e^{[\cdot]} : \mathcal{T}_{\mathrm{Q}}^{\infty} \to \mathbb{R}^+ \tag{10.2}$$

is a generalized cross ratio on the ideal boundary of a tree. Whether the cross ratio $e^{[\cdot]}$ on $\mathcal{T}_{\mathrm{Q}}^{\infty}$ is analogous to the cross ratio of the length cocycle of the metric $\widetilde{g}$ remains an open question for future research.

Concerning the question about Hölder continuity, it is not difficult to see that $[\cdot]$ is a continuous function on $\mathcal{T}_{\mathrm{Q}}^{\infty}$ endowed with the relative topology of the product topology of the standard topology for $\mathcal{T}(\infty)$ introduced in [Wei04]: the rays from a bifurcation of a quadrilateral to the points of the quadrilateral are finally in distinct branches of the tree, so they can be altered each one in an open set of $\mathcal{T}(\infty)$ without changing any of the bifurcations. Regarding the Hölder continuity, in a future work, one could try to find Lipschitz estimates when an appropriate metric on $\mathcal{T}_{\mathrm{Q}}^{\infty}$ was found.

10.3 Concordance as a Symmetric 2-Form?

In Section 8.3 we introduced a unit tangent bundle $\mathrm{T}^1\mathcal{T}$ to a tree $\mathcal{T}$ as

$$\mathrm{T}^1\mathcal{T} = \left\{ \quad (z,l) \quad : \quad l \in \mathrm{T}^1_z\mathcal{T}, \quad z \in \mathrm{V}(\mathcal{T}) \quad \right\}.$$

The unit tangent space

$$\mathrm{T}^1_z\mathcal{T} = \left\{ \quad (\eta,\xi) \in \mathcal{V}_\mathcal{T} \quad : \quad [z,\eta,z,\xi] = 0 \quad \right\}$$

at a vertex z consists of all possible velocities $v \in \mathcal{V}_\mathcal{T}$ that a geodesic through z may have.

Compared to the analysis of differentiable manifolds, where bilinear forms on the tangent spaces play an important role, a natural question to ask is, whether there is some kind of "bilinear form" on $\mathrm{T}^1_z\mathcal{T}$ or rather on an extension of $\mathrm{T}^1_z\mathcal{T}$ to some $\mathbb{Z}$-module. How such an extension could be constructed must be left as an open question for future research.

However, a minimum requirement to such a "bilinear form" $\langle\cdot,\cdot\rangle$, since linear, would be the feature that it changes sign under inversion in one argument,

$$\langle -v, w\rangle = -\langle v, w\rangle$$

for $v, w \in \mathrm{T}^1_z\mathcal{T}$. For differentiable manifolds, corresponding to a change in sign of a tangent vector is the change of direction of a curve representing that vector. For trees, this operation is expressed in terms of a velocity $(\eta,\xi) \in \mathcal{V}_\mathcal{T}$ by

$$-(\eta,\xi) := (\xi,\eta).$$

Since we are interested in objects that are sustained by a projection to a quotient of the tree, we ask for invariance of $\langle\cdot,\cdot\rangle$ under some group H of isometries. So we are to find a function $F : \mathcal{T}_{\mathbb{Q}}^{\infty} \to \mathbb{R}$ such that

$$h * F = F \quad \text{for all} \quad h \in H \quad \text{and} \quad (1\,2) * F = -F.$$

To determine all functions with these properties for a given tree and a group $H < \mathrm{Aut}\,(\mathcal{T})$ seems a highly non-trivial problem. Yet, tools are prepared to determine all "bilinear forms" in the space `UI` that was introduced in Section 9.1.

10.2 Proposition. *The functions $F \in$ `UI` that change sign under the permutation $(1\,2)$ consist exactly of the subspace spanned by the concordance $[\cdot]$.*

Proof. This is clear if $\dim(\mathtt{UI}) = 0$. Otherwise $\Phi(1\,2)$ corresponds to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to the base `del`, $[\cdot]$, `diam` by Proposition 9.9. Now the equation $(1\,2)*F = -F$ is by definition (9.2) equivalent to the equation $\Phi(1,2)(F) = -F$ or $\big(\Phi(1\,2) + \mathrm{Id}|_{\mathtt{UI}}\big)(F) = 0$. This equation is expressed in coordinates by

$$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The solution is $\mathbb{C}x_2$ and corresponds to the functions $\mathbb{C}[\cdot]$. □

Like all functions of `UI`, the concordance is by Proposition 9.14 invariant under the Klein 4-group V, which implies for all quadrilaterals $(\alpha, \omega, \eta, \xi)$ of boundary points that

$$[\eta, \xi, \alpha, \omega] = (1\,3)(2\,4) * [\alpha, \omega, \eta, \xi] = [\alpha, \omega, \eta, \xi].$$

The concordance would be a "symmetric bilinear form".

Appendix A

Groups

For the following selection of group-related topics we refer to [KS98].

A.1 Definition (Klein 4-group)**.** The Klein 4-group consists of four elements $\{\mathrm{Id}, a, b, c\}$ that obey multiplication rules as made precise in Figure A.1. As the reader can see, the Klein 4-group is abelian and all non-trivial element are involutions.

	Id	a	b	c
Id	Id	a	b	c
a	a	Id	c	b
b	b	c	Id	a
c	c	b	a	Id

Figure A.1: The group table of the Klein 4-group.

As an explicit calculation shows, the Klein 4-group is isomorphic to the normal subgroup

$$V := \big\{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3), \mathrm{Id}\big\}$$

of the symmetric group S_4. We refer henceforth with the name Klein 4-group to the group $V < S_4$.

A.2 Proposition. *The group S_4 acts by conjugation on V:*

$$(\sigma, \tau) \longmapsto \alpha\big(\sigma\big)(\tau) := \sigma\tau\sigma^{-1}$$

for all $\sigma \in S_4, \tau \in V$. *For all non-trivial elements* $(i\,j)(k\,l) \in V$ *holds* $\alpha(\sigma)\big((i\,j)(k\,l)\big) = \big(\sigma(i)\,\sigma(j)\big)\big(\sigma(k)\,\sigma(l)\big)$.

Proof. The group S_4 acts on S_4 by conjugation. Two permutations $\tau_1, \tau_2 \in S_4$ are conjugate in S_4 if and only if they have the same cycle structure. Since the group elements $(1\,2)(3\,4), (1\,3)(2\,4)$ and $(1\,4)(2\,3)$ are the only elements of S_4 that are composed of two cycles of length two, V is a normal subgroup and therefore the action of S_4 on S_4 restricts to an action on V.

Further, for all permutations $\sigma \in S_4$ and for all cycles $(i\,j) \in S_4$ holds the relation $\sigma(i\,j)\sigma^{-1} = \big(\sigma(i)\,\sigma(j)\big)$. Thus for $(i\,j)(k\,l) \in V$ holds $\sigma(i\,j)(k\,l)\sigma^{-1} = \sigma(i\,j)\sigma\sigma^{-1}(k\,l)\sigma^{-1} = \big(\sigma(i)\,\sigma(j)\big)\big(\sigma(k)\,\sigma(l)\big)$. □

The *dihedral group* D_{2n} for $n \in \mathbb{N}$ can be defined as a group generated by two involutions g, h such that the product gh has order n.

A.3 Proposition. *For the group action* $\alpha : S_4 \to \mathrm{Aut}\,(V)$ *(*Prop. A.2*) holds* $\alpha(S_4) \simeq D_6 \simeq S_3$ *and* $\ker(\alpha) = V$.

Proof. The Klein 4-group V is abelian hence $V \subset \ker(\alpha)$. We show that $\alpha(S_4) \simeq D_{2n}$ for $n = 3$. The group $\alpha(S_4)$ is generated by $\alpha(\sigma)$ and $\alpha(\tau)$ for $\sigma := (1\,2\,3\,4)$ and $\tau := (1\,2)$. The group elements $\alpha(\sigma)$ and $\alpha(\tau)$ are involutions because $\sigma^2 \in V \subset \ker(\alpha)$ and $\tau^2 = \mathrm{Id}_{S_4}$. The group elements $\alpha(\sigma)$ and $\alpha(\tau)$ are not trivial: $\alpha(\sigma)\big((1\,2)(3\,4)\big) = (1\,4)(2\,3)$ and $\alpha(\tau)\big((1\,4)(2\,3)\big) = (1\,3)(2\,4)$. Since $\tau\sigma = (2\,3\,4)$ this shows also that $\alpha(\tau)\alpha(\sigma)$ has order three. Thus $\alpha(S_4) \simeq D_6$. By Lagrange follows now $\ker(\alpha) = V$. □

A.4 Proposition. *If* F *is a group and two homomorphisms* $\alpha : F \to G$ *and* $\beta : F \to H$ *for groups* G *and* H *are given such that* $\ker(\alpha) \subset \ker(\beta)$ *then there is a homomorphism* $\gamma : G \to H$ *such that* $\gamma \circ \alpha = \beta$.

Proof. By the first Isomorphism Theorem one has $G \simeq F/\ker(\alpha)$. Thus a map $\gamma : G \to H$ is well defined by $\gamma\big(g\ker(\alpha)\big) := \beta(g)$ for $g\ker(\alpha) \in G$. Since $\ker(\alpha)$ is a normal subgroup of F the map γ is a homomorphism. Finally observe that for all $g \in F$ holds $\gamma \circ \alpha(g) = \gamma\big(\alpha(g)\big) = \gamma\big(g\ker(\alpha)\big) = \beta(g)$. □

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Erklärung des Autors

Hiermit erkläre ich, dass ich meine Diplomarbeit selbstständig und nur unter Verwendung der angegebenen Literatur angefertigt habe.

Erlangen, 19. Juli 2004